Cantor Set and Its Properties

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Abstract

The Cantor set is a famous set first introduced by German mathematician Georg Cantor in 1883. It is simply a subset of the interval $[0, 1]$, but it has a number of remarkable and deep properties. We will first describe the construction and the formula of the Cantor ternary set, which is the most common modern construction, and then prove some interesting properties of the set.

Preliminaries

Definition: If A and B are sets, the *union* of A and B, written $A \cup B$, is the set of all objects that belong to either A or B or both. $A \cup B = \{x : x \in A \text{ or } x \in B\}$

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Example:
Let $A = \{1, 2, 3\}, B = \{2, 3, 4\}$. Then

$A \cap B = \{2, 3\}, A \cup B = \{1, 2, 3, 4\}$

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Example:
Let $A_n = \{n\}$ for each $n \in \mathbb{N}$. Then

$$\bigcap_{n=1}^{\infty} A_n = \emptyset$$

$$\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$$

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Next, the open middle third of each of these remaining segments is deleted, leaving four line segments: $\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.

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This process is continued to infinity.

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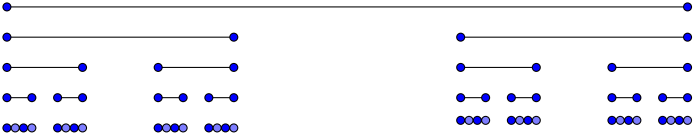

Formula

Consider the following set:

$$C = \bigcap_{n=1}^{\infty} \bigcap_{k=0}^{3^{n-1}-1} \left(\left[0, \frac{3k+1}{3^n}\right] \cup \left[\frac{3k+2}{3^n}, 1\right] \right)$$

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Another explicit formula for Cantor set is

$$C = [0, 1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right)$$

Properties and Proofs

Now we will prove some interesting properties of C.

Property 1

Let $x = 0.a_1 a_2 a_3 \ldots$ be the base 3 expansion of a number $x \in [0, 1]$. Then $x \in C$ iff $a_n \in \{0, 2\}$ for all $n \in \mathbb{N}$

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Before we prove property 1, we first have a look at base 3 expansion of a number.

Fractions in base b

A fraction N in base b is represented in terms of the negative powers of b:
$N = a_1 b^{-1} + a_2 b^{-2} + a_3 b^{-3} + ... + a_n b^{-n}$ $(\forall i \in [0, n], a_i \in (0, b))$

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How do we convert $0.a_1 a_2 a_3..._{10}$ to another base?

(1) First, pick up the coefficients $a_1, a_2, ...a_n$.
(2) Multiply $0.a_1 a_2 a_3..._{10}$ by b.
(3) The integer part of the result is a_1.
(4) If the remaining part is zero, stop.
(5) Otherwise, let $m = 2$.
(6) Multiply the remaining part by b.
(7) The integer part of the result is a_m. If the remaining part is zero, stop.
(8) Otherwise, let $m = m + 1$ and goto step (6).

Example:

Convert 0.375_{10} to base 2

$2N = 2 \times 0.375 = 0.75 \longrightarrow a_1 = 0,\ r_1 = 0.75 - \lfloor 0.75 \rfloor = 0.75$

$2r_1 = 2 \times 0.75 = 1.5 \longrightarrow a_2 = 1,\ r_2 = 1.5 - \lfloor 1.5 \rfloor = 0.5$

$2r_2 = 2 \times 0.5 = 1 \longrightarrow a_3 = 1,\ r_3 = 1 - \lfloor 1 \rfloor = 0$

Since $r_3 = 0$, we stop. $0.375_{10} = 0.011_2$

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For example, for the number 0.201_3, 2 means that it is in the third third of $[0, 1]$, 0 means that it is in the first third of $\left[\frac{2}{3}, 1\right]$ (the third third of $[0, 1]$), and 1 means that it is in the second third of $\left[\frac{2}{3}, \frac{5}{9}\right]$ (the first third of $\left[\frac{2}{3}, 1\right]$).

Assume that there exists some $k \in \mathbb{N}$ such that $a_k = 1$, then x will be in the middle third of some interval whose middle third will be removed, which means that $x \notin C$

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On the other hand, by the definition of base 3 expansion, if $a_n \in \{0, 2\}$ for all $n \in \mathbb{N}$, x will never be in the middle third of any interval whose middle third will be removed. Thus, $x \in C$.

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The **cardinality** of the domain of a surjective function is greater than or equal to the cardinality of its codomain, that is, if $f : X \to Y$ is a surjective function, then X has at least as many elements as Y.

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From property 1, we have that for any $x = 0.a_1a_2..._3 \in [0, 1]$, $x \in C$ iff $a_n \in \{0, 2\}$ for all $n \in \mathbb{N}$

Then we construct a function $f : C \to [0, 1]$ which replaces all the 2s by 1s, and interprets the sequence as a binary representation of a real number. In a formula,

$$f\left(\sum_{k=1}^{\infty} a_k 3^{-k}\right) = \sum_{k=1}^{\infty} \frac{a_k 2^{-k}}{2}$$

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For any number y in [0,1], its binary representation can be translated into a ternary representation of a number x in C by replacing all the 1s by 2s, so the range of f is $[0, 1]$. Thus, the cardinality of C is greater than or equal to the cardinality of $[0, 1]$, which means that C is uncountable.

Property 3

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The sum of the length of all intervals removed is

$$\sum_{n=1}^{\infty} 2^{n-1}\left(\frac{1}{3^n}\right) = \frac{1}{3}\sum_{n=0}^{\infty} \frac{2}{3}^{n} = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 1$$

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Thus, the length of the complement of C relative to $[0, 1]$ is 1, which means that C has a length of zero.

Thank you

Thank you all for listening to my presentation.

Questions?