Fair Division Algorithms

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June 4, 2014

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- One way is to divide the cake in half and give each of you one of the halves.

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- Is there a solution to divide this cake so that each of you guaranteed at least fair share of this cake?

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- This algorithm: guarantees the first play who cuts the cake get at least half of the value of the cake. And the second player who is the first one to choose get at least half of the value of the cake because he can choose from two pieces and one of which must be half of the value of the cake or larger in his eyes.

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- When there are only two participants, the algorithm is both proportional and envy-free.

For 3 players

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- Step 3: p_2 then choose one of the remaining two pieces.
- Step 4: p_1 gets the remaining one.

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- Step n: The last player gets the last piece of cake.

Lone Chooser Protocol

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- Step 1: p_1 and p_2 play cut-and-choose protocol and p_1 cut the cake. And p_1 gets S_1, p_2 gets S_2. $X = S_1 \cup S_2$ and $v_1(S_1) = \frac{1}{2}$ and $v_2(S_2) \geq \frac{1}{2}$.

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  p_1 cuts S_1 into 3 pieces: S_{11}, S_{12}, S_{13} such that
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  p_2 cuts S_2 into 3 pieces: S_{21}, S_{22}, S_{23} such that
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  p_2 cuts S_2 into 3 pieces: S_{21}, S_{22}, S_{23} such that
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  p_3 choose one piece he thinks is the most valuable from $\{S_{11}, S_{12}, S_{13}\}$ and one piece he thinks is the most valuable from $\{S_{21}, S_{22}, S_{23}\}$

Lone Chooser Protocol

- Step n-1: Player $p_i, 1 \leq i - 1$ has a couple of pieces of cake P_i with $v_i(P_i) \geq \frac{1}{n-1}$. Then cuts P_i into n pieces $P_{i1}, P_{i2}, ..., P_{in}$ such that

$$v_i(P_{i1}) = v_i(P_{i2}) = v_i(P_{i3}) = \cdots = v_i(P_{in}) \geq \frac{1}{n(n-1)}$$

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- Player P_n then chooses from P_{i1} through P_{in} for $i, 1 \leq i \leq n-1$ that is most valuable according v_n.

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- Notice that the cut-and-choose algorithm is both proportional and envy-free.
- Review the defination: Proportionality means that every player has value at least $\frac{1}{n}$ for its piece of cake. Envy-freeness means that every player weakly prefers his own piece to any other piece.
- It's very hard to design a envy-free algorithm for 3 or more players. Can you design one envy-free algorithm for 3 players?

Why we care

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- But people do care about making sure that he or she gets the fair share of an inheritance.
- Divide land, Divide time, Minimize the burden of chores.

Sourses

1 http://ccc.cs.uni-duesseldorf.de/ rothe/CAKE/folien-3-protocols.pdf

2 R. Simpson, M. Logan, P. Dolan. Fair Division Algorithms. 2013

3 Ariel D. Procaccia. Cake Cutting Algorithms