Finding Perfect Matchings and Completing Latin Rectangles

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Outline

- Definitions

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- The Problem

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- The Set up

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- The Process

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- The degree of a vertex in a graph is the number of edges that vertex is connected to

Bipartite Graph

Definition: A Bipartite Graph is a graph such that every vertex can be put into one of two groups, V_1 and V_2, with the property that all edges connect vertices in V_1 to vertices in V_2 (no edges go between vertices in the same group)

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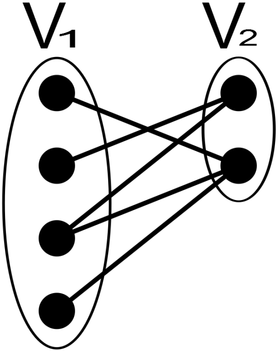
Example:



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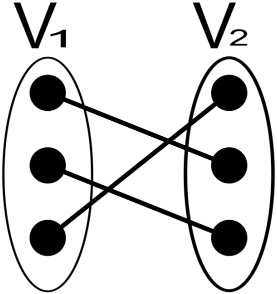

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- How can we find these Perfect Matchings?

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- And label the vertices in V_2 the numbers 1, 2,...,(n-1), n

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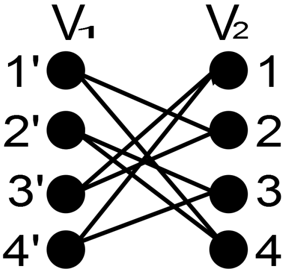

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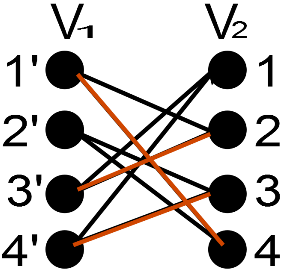
- This is notably not a perfect matching, and we cannot easily make it one because there does not exist an edge in our initial graph from 2' to 1



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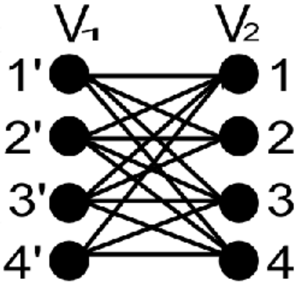

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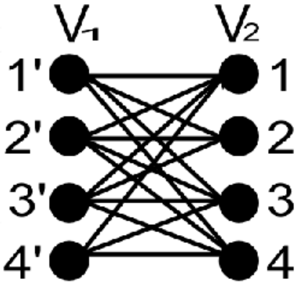

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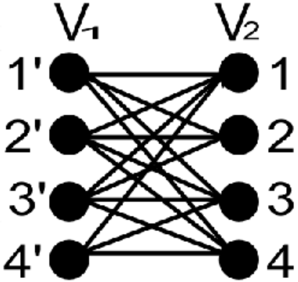

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- Each column is headed by a vertex in V_1
- Each other number in the column is a vertex in V_2 such that there is an edge between the head of a column and each element in its column

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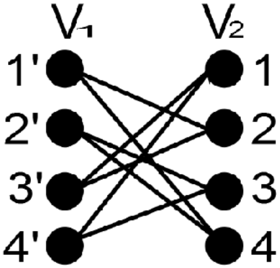

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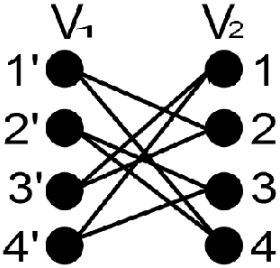

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2	X_2	2	X_2
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For edges that we removed from the d=n graph, we put an X in the array to indicate that there is not an edge

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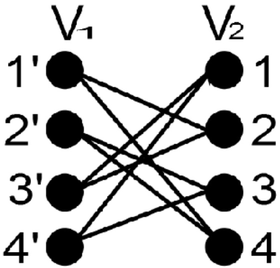
To best understand this process, let us find a perfect matching of the previous example:



$\Longrightarrow$

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As $K_c = 2$ for all the columns we can pick a random element from any of the columns, let us then arbitrarily pick column 2' element 3

The Process

Thus having picked (2',3) we shall make all other elements in column 2' and row 3 X:

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The Process

There are now three rows left. $K_1 = K_3 = 2$ while $K_4 = 1$ thus we must pick from column 4' the element 1

$1'$	$2'$	$3'$	$4'$
X_1	X_1	1	1
2	X_2	2	X_2
X_3	3	X_3	X_3
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Having picked (4',1) we must now X out all other elements in Column 4' and row 1:

$1'$	$2'$	$3'$	$4'$
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The Process

We now have that $K_1 = 2$ and $K_3 = 1$ so we must pick from column 3' element 2:

$1'$	$2'$	$3'$	$4'$
X_1	X_1	X_1	1
2	X_2	2	X_2
X_3	3	X_3	X_3
4	X_4	X_4	X_4

The Process

Putting X's in the appropriate places yields:

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2	X_2	2	X_2
X_3	3	X_3	X_3
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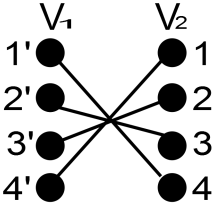
Leaving only element (1',4) to pick. The edges defined by (1',4), (2',3), (3', 2), and (4',1) thus form a perfect matching:



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- Because we must pick from the column with the fewest elements to pick from, the only reason we would not have select (A'_0, e_0) would be if there were another column, A'_1, that also only had one element, e_1, left to pick
- And specifically the only reason (A'_1, e_1) would prevent (A'_0, e_0) from getting chosen is if $e_1 = e_0$ so as to then change e_0 to an X

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- For A_2' to be selected and leave behind the situation we had above (going one step back from $K_{A_1} = 0$) we get that all three columns must've shared an element that was selected in column A_2', and we shall call this element e_2

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- This means that the process detailed in this presentation will always yield a perfect matching

Applications

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Example of a 5 by 5 Latin Square:

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Bipartite Graph Features	Analogous Features of Latin Squares
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Bipartite Graph Features	Analogous Features of Latin Squares
Each vertex in V_1	One specific Column
Each vertex in V_2	One specific symbol
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As we can create d perfect matchings of a Bipartite Graph, we can complete the Latin Rectangle

Thank you

Thank You all for listening to this presentation
Questions?