Curvature

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When drawing two circles of different radii it certainly seems like the smaller one is curving more rapidly than the the larger one.

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So how can we incorporate these ideas?

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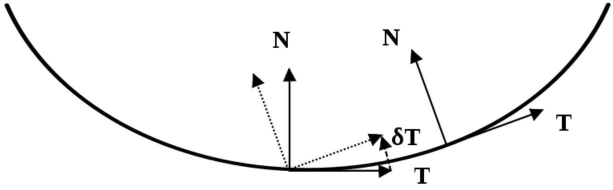

Tangent.png

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T is a vector having both magnitude ans direction. As the magnitude of T isn't changing we are really just measuring the change the direction, or angle with respect to the horizontal T is making, which we shall call θ, so alternatively this we can say
$K = ||\frac{d\theta}{ds}||$

If we are working with a standard defined function in the x-y plane, y=f(x), then we can use the chain rule to write
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As we know that $tan(\theta) = \frac{dy}{dx}$ and that $\frac{dx}{ds} = \frac{1}{\sqrt{1+(\frac{dy}{dx})^2}}$ we can thus use some messy but basic calculus to reduce this all to

$$K = ||\frac{\frac{d^2y}{dx^2}}{[1+(\frac{dy}{dx})^2]^{3/2}}||$$

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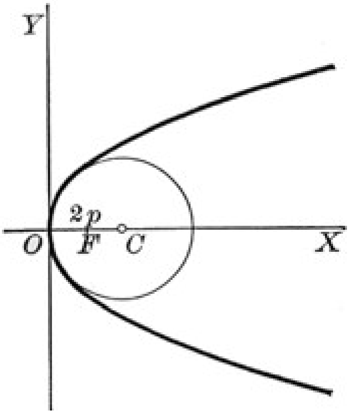
But one thing we can see is that at x=0 K=2, and recalling from the beginning that **radius of curvature** is the inverse of curvature, we get that the bottom a standard parabola behaves momentarily like a circle of radius 1/2!


Y
2p
O
F
C
X

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The intersection of the normal plane and our surface is a two dimensional curve, from which in the manner described in the rest of this presentation we can find a value for the curvature.

Different normal planes will have different values for the curvature at the specific point

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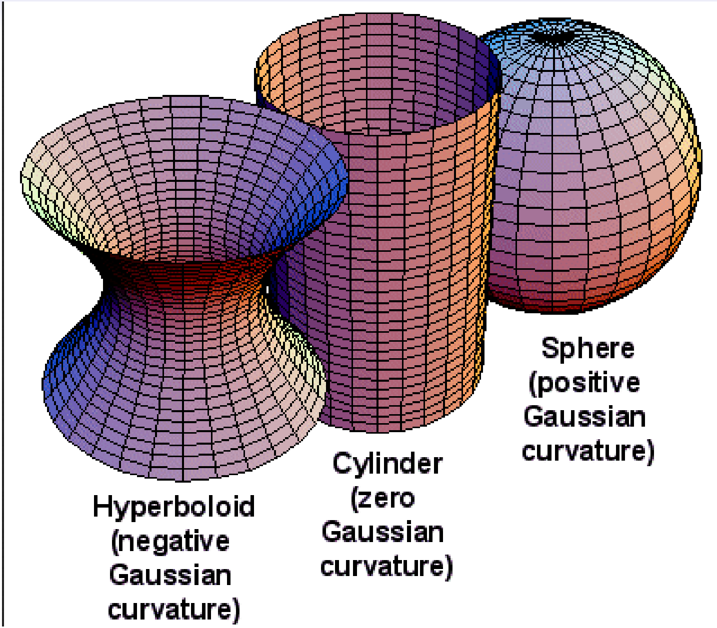
When we use this definition we get that the following:


Hyperboloid
(negative
Gaussian
curvature)
Cylinder
(zero
Gaussian
curvature)
Sphere
(positive
Gaussian
curvature)

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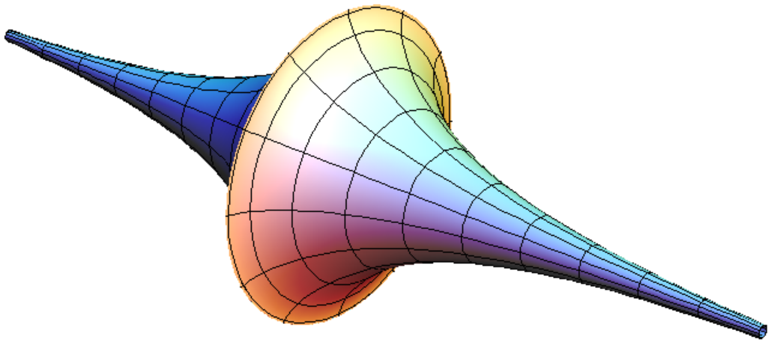
While this is conceptually harder to understand, others have figure out that it takes the form of the following:



Thank you! Questions?

Sources: Wikipedia, Wolfram Alpha, and Google images (I looked at other places but these had the nicest things to read and understand to for a beginner)