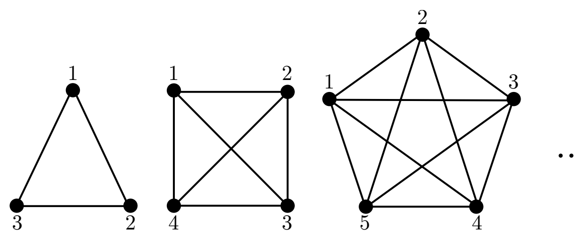
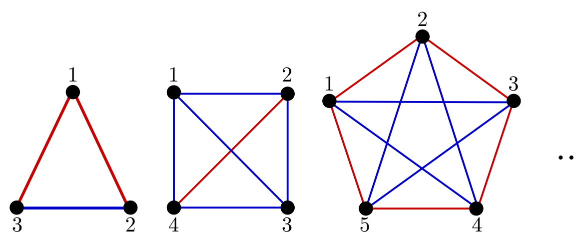
Math/CCS 103	Professor: Padraic Bartlett
Mini-presentation : Ramsey Theory	
Week 1	*UCSB 2014*

This miniature talk builds off of the game of Sim we mentioned earlier! Consider the following definitions:

1. The complete graph on n vertices, K_n, is the simple graph on the vertex set $\{v_1, v_2, \ldots v_n\}$ that has every possible edge: in other words, $E(K_n) = \{\{v_i, v_j\} : i \neq j\}$. We draw several of these graphs below:



2. A **two-coloring** of the edges in a graph, given two colors (say, red and blue,) is a mapping that assigns to each edge in the graph one of these two colors.



Game. Consider the following game, which is the n, k-generalization of Sim:

Board: n vertices drawn on a piece of paper.

Players: Two players, Red and Blue, each with a pen of their respective color.

Play: The players alternate turns, drawing edges between vertices using their colored pens. Edges must be drawn between vertices that have not been connected with an edge yet; i.e. once one player connects two vertices with an edge, no other player can draw an edge that connects those two vertices.

End state: A player loses if there is ever a collection of k vertices connected only by edges of that player's color: i.e. if that player creates a monochromatic K_k in their color.

In other words, we are playing on a K_n, and each player is trying to avoid making a monochrome K_k in their respective color: in Sim, we were playing on a K_6, and each player was trying to avoid making a K_3.

In this presentation, we will prove the following theorem:

Theorem 1. *For any k, there is a n such that games of (n, k)-Sim never end in draws.*

In particular, we will prove the following stronger statement:

Theorem 2. *For any two integers k, l, there is a n such that any red-blue coloring of K_n contains either a red K_k or a blue K_l.*

Proof. Let $R(k, l)$ denote the smallest value of n such that if K_n's edges are all colored either red or blue, then K_n necessarily contains an all-red K_k or an all-blue K_l. We seek to show that R is well-defined, and always exists.

We first note some simple starting cases. We have $R(n, 1) = R(1, n) = 1$, as any two-coloring of K_n's edges has a K_1 in which all of the edges are whatever color we want (because there are no edges in K_1, as it is the graph with one vertex and no edges.)

As well, we have $R(n, 2) = R(2, n) = n$, because any red-blue two-coloring of K_n's edges either

- paints all of the edges the same color (which makes a monochrome K_n of one of our colors), or
- paints at least one edge red and another blue (which makes monochrome K_2's of both colors.)

Furthermore, we claim that we have the following recursive bound on the growth of $R(r, s)$:

$$R(r, s) \leq R(r, s-1) + R(r-1, s)$$

To prove this, we proceed by induction on the sum $r + s$. We've already proven the base cases via the two examples above: so we take any pair r, s, and can assume that our bound holds for any x, y with $x + y < r + s$.

Take a complete graph K on $(R(r, s-1) + R(r-1, s))$ many vertices, and color its edges red and blue. We seek to show that there's either a monochrome red K_r or monochrome blue K_s in K_n.

To see this, we mimic the proof structure that worked for us in our game. Pick any $v \in K$, and partition the rest of K's vertices into two sets:

- B', which contains all of the vertices in K connected to v by a blue edge, and
- R', which contains all of the vertices in K connected to v by a red edge.

Let B and R be the subgraphs[1] of K induced by these vertices, respectively.

Because K has

$$R(r, s-1) + R(r-1, s) = |V(B)| + |V(R)| + 1$$

many vertices, either $|V(B)| \geq R(r, s-1)$ or $|V(R)| \geq R(r-1, s)$.

Suppose that we have $|V(B)| \geq R(r, s-1)$. Because $r + s - 1 < r + s$, we can apply our inductive hypothesis, which tells us that we have either

[1]Given a graph $G = (V, E)$ and a subset of vertices $X \subset V$ from G, the **subgraph induced by** X is the graph with vertex set X, where two vertices are connected in X whenever they are connected in G.

1. a red K_r inside of B, or

2. a blue K_{s-1} inside of B, in which case (by combining this blue K_{s-1} with v and its edges to B) we have a blue K_s inside of our entire K_n.

These are the two cases we were looking for; so, in the situation where $|V(B)| \geq R(r, s-1)$, we've proven our claim!

Similarly, if we have $|R| \geq R(r-1, s)$, we can use induction to tell us that there's either

1. a blue K_s inside of R, or

2. a red K_{r-1} inside of R, in which case (by combining this red K_{r-1} with v and its edges to R) we have a red K_r inside of our entire K_n,

and we're also done.

□

In the language of the proof above, our Sim presentation can be thought of as proving that $R(3, 3) = 6$.