Math/CCS 103	Professor: Padraic Bartlett

Mini-presentation : Sim

Week 1 — *UCSB 2014*

Consider the following game:

Game. This is the game of Sim. It is played as follows:

Board: Six vertices drawn on a piece of paper.

Players: Two players, Red and Blue, each with a pen of their respective color.

Play: The players alternate turns, drawing edges between vertices using their colored pens. Edges must be drawn between vertices that have not been connected with an edge yet; i.e. once one player connects two vertices with an edge, no other player can draw an edge that connects those two vertices.

End state: A player loses if there is ever a collection of three vertices connected only by edges of that player's color: i.e. if that player creates a monochromatic triangle in their color.

In this brief presentation, we will prove the following theorem:

Theorem 1. *This game never ends in a draw.*

Proof. To see why, we proceed by contradiction: suppose not! Then there is some board on which no monochromatic triangles exist. In the language of graph theory, this is a graph on six vertices in which we've drawn every possible edge between vertices, and given each edge a color, either red or blue.

Take any vertex v in this graph. Because there are five other vertices in our graph, the degree of our vertex $deg(v)$ — i.e. the number of edges leaving this vertex — is 5. Therefore, if we know that these five edges are shaded red and blue, there must be at least three of these edges that are the same color! Suppose (without any loss of generality) that this color is red, and let $\{w_1, w_2, w_3\}$ be the endpoints of these edges.

Then, there are two cases:

- There is some edge $\{w_i, w_j\}$ that's red. In this case, the vertices v, w_i, w_j form a red triangle.
- Every edge $\{w_i, w_j\}$ is blue. In this case, the vertices w_1, w_2, w_3 form a blue triangle.

In either situation, we've found a monochrome triangle! So these always exist, and therefore we have a contradiction. $\square$

This is part of a more general area of mathematics, called Ramsey theory. Roughly speaking, Ramsey theory is the study of "how much order there must be in any given structure." For example, we have just proven that in any 2-coloring of the edges of K_6 (the graph on 6 vertices that contains every possible edge,), there must be a monochrome K_3 (i.e. a triangle!) Ramsey's theorem proves that given any K_k, there is some sufficiently large value of n such that no matter how you two-color the edges of a K_n, there must be a monochrome K_k.

Finding these values is a remarkably wide-open problem: i.e. we still do not know the smallest value of n such that a K_5 is forced to exist in any two-coloring of the edges of K_n. In future lectures, we will explore this theory further!