| Math/CCS 103 | Professor: Padraic Bartlett |
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# Lecture 11: The Unit Distance Graph

*Week 8* — *UCSB 2014*

(Source: "The Mathematical Coloring Book," by Alexander Soifer.)

## 1 The Unit Distance Graph Problem

**Definition.** Consider the following method for turning $\mathbb{R}^2$ into a graph:

- Vertices: all points in $\mathbb{R}^2$.
- Edges: connect any two points $(a, b)$ and $(c, d)$ iff the distance between them is exactly 1.

This graph is called the **unit distance graph.**

Visualizing this is kinda tricky — it's got an absolutely insane number of vertices and edges. However, we can ask a question about it:

**Question.** How many colors do we need in order to create a proper coloring of the unit distance graph?

So: the answer isn't immediately obvious (right?) Instead, what we're going to try to do is just **bound** the possible answers, to get an idea of what the answers might be.

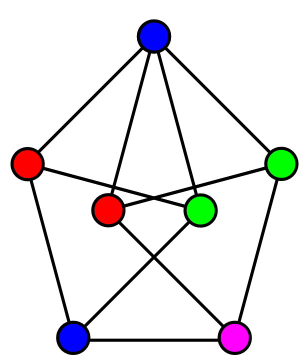
How can we even bound such a thing? Well: to get a lower bound, it suffices to consider finite graphs $G$ that we can draw in the plane using only straight edges of length 1. Because our graph on $\mathbb{R}^2$ must contain any such graph "inside" of itself, examining these graphs will give us some easy lower bounds!

So, by examining a equilateral triangle $T$, which has $\chi(T) = 3$, we can see that

$$\chi(\mathbb{R}^2) \geq 3.$$

This is because it takes three colors to color an equilateral triangle's vertices in such a way that no edge has two endpoints of the same color.

Similarly, by examining the following pentagonal construction (called a Moser spindle,)



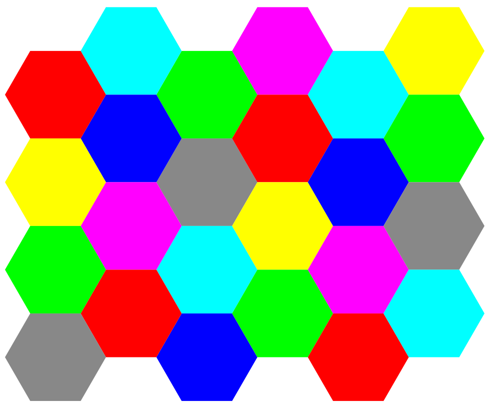
we can actually do one better and say that

$$\chi(\mathbb{R}^2) \geq 4.$$

Verify for yourself that you can't color this graph with three colors!

Conversely: to exhibit an upper bound on $\chi(\mathbb{R}^2)$ of $k$, it suffices to create a way of "painting" the plane with $k$-colors in such a way that no two points distance 1 apart get the same color.

So: consider the following way to color the plane!



To be specific: start by tiling the plane with hexagons of diameter slightly less than 1. Then, color the hexagons with seven colors as described above; i.e. repeat the color pattern

gray, red, teal, yellow, blue, green, magenta

on each strip of hexagons, shifted two colors over for each strip. This gives you a mesh of hexagons, so that any two hexagons of the same color are at least more than distance 1 apart. Therefore, any line segment of length 1 cannot bridge two different hexagons of the same color! As well, because the hexagons have diameter slightly less than one, no line segment of length 1 can lie entirely within a hexagon of the same color. Therefore, there are no line segments of length 1 with both endpoints of the same color!

In other words, we have just proven that this is a proper coloring of the plane! So we can color the plane with seven colors: i.e. we just showed that

$$\chi(\mathbb{R}^2) \leq 7.$$

These bounds on $\chi(\mathbb{R}^2)$ were not too crazy to find: it took us no more than two pages to get here, starting from the basic definition of a graph! As a result, we might hope that completely resolving this question is something we could easily finish within a few more pages.

Surprisingly: the answer is no! This problem – often called the Hadwiger-Nelson problem in graph theory literature – has withstood attacks from the best minds in combinatorics since the 1950's, and is still open to this day.

So: it's not too likely that we're going to be able to solve this problem in this class. (Try it, though!) If we were going to try, though, how would we attempt to come up with a solution?

Typically, when presented with an open or difficult problem, mathematicians rarely attempt to **directly** solve the problem; if this was likely to succeed, someone probably would have done it already! Instead, what we do is try to create a related problem to the one we want to study; we either take a special case of the original problem, or remove some conditions from it, or attempt to get a weaker conclusion, or other such things.

For the unit distance graph problem, a natural question to ask is the following: if we can't find the chromatic number of $\mathbb{R}^2$, maybe we can find the chromatic number of other spaces! For example, consider the following vertex sets, which we turn into graphs by connecting all vertices at distance 1:

1. $\mathbb{R}^1$: i..e the real line! This object has chromatic number 2; simply color the points in intervals of the form $[2k, 2k+1)$ red, and points in intervals of the form $[2k+1, 2k+2)$ blue. Any two points at distance 1 cannot lie in the same interval, nor can they span two nonadjacent intervals, so this is a proper 2-coloring!

2. $\mathbb{Z}^k$, for any $k$: i.e. the integer lattice! This also has chromatic number 2: simply color every point of the form $(x_1, \ldots x_k)$ with $\sum x_i =$ even red, and every point with $\sum x_i =$ odd blue. Any two points that are distance 1 apart are identical in all but one of their coordinates, at which they differ by 1; therefore those two points must be different colors, and thus we have a 2-coloring.

3. $\mathbb{R}^3$: i.e. three-dimensional space! This turns out to be harder than the unit distance problem. Using arguments similar to the ones we used for two-dimensional space, we can see that $6 \leq \chi(\mathbb{R}^3) \leq 15$; try to check this out on your own!

4. $\mathbb{Q}^2$: i.e. the rational plane! On one hand, this seems like it should be similar to $\mathbb{R}^2$: they're both dense two-dimensional spaces full of points that seem tricky to color! On the other hand, unlike $\mathbb{R}^2$, we cannot construct things like equilateral triangles; so it is not obvious how to even get a lower bound of 3. To work on this, we need the following concept:

## 2 Equivalence Relations

**Definition.** Take any set $S$. A **relation** $R$ on this set $S$ is a map that takes in ordered pairs of elements of $S$, and outputs either true or false for each ordered pair.

You know many examples of relations:

- Equality (=), on any set you want, is a relation; it says that $x = x$ is true for any x, and that $x = y$ is false whenever $x$ and $y$ are not the same objects from our set.

- "Mod $n$" ($\equiv \mod n$) is a relation on the integers: we say that $x \equiv y \mod n$ is true whenever $x - y$ is a multiple of $n$, and say that it is false otherwise.

- "Less than" (<) is a relation on many sets, for example the real numbers; we say that $x < y$ is true whenever $x$ is a smaller number than $y$ (i.e. when $y - x$ is positive,) and say that it is false otherwise.

- "Beats" is a relation on the three symbols (rock, paper, scissors) in the game Rock-Paper-Scissors. It says that the three statements "Rock beats scissors," "Scissors beats paper," and "Paper beats rock" are all true, and that all of the other pairings of these symbols are false.

In this class, we will study a specific class of particularly nice relations, called **equivalence relations**:

**Definition.** A relation $R$ on a set $S$ is called an **equivalence relation** if it satisfies the following three properties:

- **Reflexivity**: for any $x \in S$, $xRx$.
- **Symmetry**: for any $x, y \in S$, if $xRy$, then $yRx$.
- **Transitivity**: for any $x, y, z \in S$, if $xRy$ and $yRz$, then $xRz$.

It is not hard to classify our example relations above into which are and are not equivalence relations:

- Equality ($=$) is an equivalence relations on any set you define it on – it trivially satisfies our three properties of reflexivity, symmetry and transitivity.
- "Mod $n$" ($\equiv \mod n$) is an equivalence relation on the integers. This is not hard to check:
  - **Reflexivity**: for any $x \in \mathbb{Z}$, $x - x = 0$ is a multiple of $n$; therefore $x \equiv x \mod n$.
  - **Symmetry**: for any $x, y \in S$, if $x \equiv y \mod n$, then $x - y$ is a multiple of $n$; consequently $y - x$ is also a multiple of $n$, and thus $y \equiv x \mod n$.
  - **Transitivity**: for any $x, y, z \in S$, if $x \equiv y \mod n$ and $y \equiv z \mod n$, then $x - y$, $y - z$ are all multiples of $n$; therefore $(x - y) + (y - z) = x - y + y - z = x - z$ is also a multiple of $n$, and thus $x \equiv z \mod n$.
- "Less than" ($<$) is not an equivalence relation on the real numbers, as it breaks reflexivity: $x \not< x$, for any $x \in \mathbb{R}$.
- "Beats" is not an equivalence relation on the three symbols (rock, paper, scissors) in the game Rock-Paper-Scissors, as it breaks symmetry: "Paper beats rock" is true, while "Rock beats paper" is false.

Equivalence relations are remarkably useful because they allow us to work with the concept of equivalence classes:

**Definition.** Take any set $S$ with an equivalence relation $R$. For any element $x \in S$, we can define the **equivalence class** corresponding to $x$ as the set

$$\{s \in S \mid sRx\}$$

Again, you have worked with lots of equivalence classes before. For $\mod 3$ arithmetic on the integers, for example, there are three possible equivalence classes for an integer to belong to:

$$\{\ldots -6, -3, 0, 3, 6 \ldots\}$$
$$\{\ldots -5, -2, 1, 4, 7 \ldots\}$$
$$\{\ldots -4, -1, 2, 5, 8 \ldots\}$$

Every element corresponds to one of these three classes.

The concept of equivalence classes is useful largely because of the following observation:

**Observation.** Take any set $S$ with an equivalence relation $R$. On one hand, every element $x$ is in some equivalence class generated by taking all of the elements equivalent to $x$, which is nonempty by reflexivity. On the other hand, any two equivalence classes must either be completely disjoint or equal, by symmetry and transitivity: if the sets $\{s \in S \mid sRx\}$ and $\{s' \in S \mid s'Ry\}$ have one element $t$ in common, then $tRx$ and $tRy$ implies, by symmetry and transitivity, that $xRy$; therefore, by transitivity, any element in one of these equivalence relations must be in the other as well.

Consequently, these equivalence classes **partition** the set $S$: i.e. if we take the collection of all distinct equivalence classes, every element of $S$ is in exactly one such set.

One particularly useful use of the concept of equivalence classes is in our definition of the rational numbers themselves! In particular, ask yourself: what is the set of the rational numbers?

Most people will quickly say something equivalent to the following:

$$\left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

The issue with this as a set is that it has **lots** of different entries for numbers that we usually think are not different objects! I.e. the set above contains

$$\frac{-1}{-2}, \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots,$$

all of which we think are the same number! People usually then go back and change our definition above to the following:

$$\left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b > 0, GCD(a, b) = 1 \right\}.$$

This fixes our issue from earlier: we no longer have "duplicated" numbers running around. However, it has other issues: suppose that you wanted to define addition on this set! Naively, you might hope that the following definition would work:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

However, for many fractions, the output of this operation is not an element of our new set!

$$\frac{2}{5}+\frac{8}{5}=\frac{40+10}{25}=\frac{50}{25}\notin\left\{\frac{a}{b}\;\middle|\; a,b\in\mathbb{Z}, b>0, GCD(a,b)=1\right\}.$$

These difficulties that we're running into with the rational numbers come from the fact that, practically speaking, **they aren't a set** in most contexts that we work with them! Rather, they are a **set with an equivalence relation**:

- The underlying set for the rational numbers: $\left\{\frac{a}{b}\;\middle|\; a,b\in\mathbb{Z}, b\neq 0\right\}$.
- The equivalence relation: we say that $\frac{a}{b}=\frac{c}{d}$ if there are a pair of integers $k,l$ such that $ka=lc$ and $kb=ld$.
- A **rational number** is any equivalence class of our set above under the above equivalence relation. This is the idea we have when we think of

  $$\frac{-1}{-2},\frac{1}{2},\frac{2}{4},\frac{3}{6},\frac{4}{8},\ldots$$

  as all representing the "same number" $1/2$: we're identifying $1/2$ with its equivalence class!
- In this setting, we define addition, multiplication, and all of our other properties just how we would normally: i.e. we define

  $$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd},$$

  where the only wrinkle is that by each of $\frac{a}{b},\frac{c}{d},\frac{ad+bc}{bd}$ we actually mean "take any element equivalent to these fractions," and by equality above we actually mean our equivalence relation.

With this detour completed, we return to our original problem:

# 3 The Chromatic Number of $\mathbb{Q}^2$

**Theorem.** The chromatic number of $\mathbb{Q}^2$ is 2.

*Proof.* Our proof proceed in a few steps, using equivalence relations. Some of the details here are left as exercises on the HW: check them out!

First, consider the following relation $\sim$ on $\mathbb{Q}^2$: for any two points $(a_1,a_2),(b_1,b_2)\in\mathbb{Q}^2$, we say that $(a_1,a_2)\sim(b_1,b_2)$ if and only the following happens: $a_1-b_1$ and $a_2-b_2$ both have odd denominators. (Because rational numbers, as discussed above, have many possible representatives, we avoid ambiguity by asking that we write these fractions with the GCD of their numerator and denominator equal to 1, and where the denominator is positive.)

For example, $\left(\frac{1}{2},\frac{1}{3}\right)\not\sim\left(\frac{3}{4},\frac{5}{6}\right)$, because their difference $\left(\frac{1}{2}-\frac{3}{4},\frac{1}{3}-\frac{5}{6}\right)=\left(\frac{-1}{4},\frac{1}{2}\right)$ does not consist of fractions whose denominators are odd. However, $\left(\frac{7}{15},\frac{4}{3}\right)\sim\left(\frac{2}{15},\frac{1}{9}\right)$, because their difference $\left(\frac{7}{15}-\frac{2}{15},\frac{4}{3}-\frac{1}{9}\right)=\left(\frac{1}{3},\frac{11}{9}\right)$ consists of fractions whose denominators are odd.

It is not hard to see that this is an equivalence relation:

- **Reflexivity**: for any $\left(\frac{a}{b}, \frac{c}{d}\right) \in \mathbb{Q}^2$, $\left(\frac{a}{b}, \frac{c}{d}\right) - \left(\frac{a}{b}, \frac{c}{d}\right) = (0,0)$. 0, when written as a fraction with the GCD of its numerator and denominator equal to 1 and with a positive denominator, has a unique representation as $\frac{0}{1}$. 1, in particular, is odd; so the difference of any point in $\mathbb{Q}^2$ with itself consists of a pair of fractions with odd denominators! Therefore any point in $\mathbb{Q}^2$ is related to itself under $\sim$: i.e. $\sim$ is reflexive.

- **Symmetry**: for any $\left(\frac{a}{b}, \frac{c}{d}\right), \left(\frac{e}{f}, \frac{g}{h}\right) \in \mathbb{Q}^2$, we want $\left(\frac{a}{b}, \frac{c}{d}\right) \sim \left(\frac{e}{f}, \frac{g}{h}\right)$ if and only if $\left(\frac{e}{f}, \frac{g}{h}\right) \sim \left(\frac{a}{b}, \frac{c}{d}\right)$. But this is trivial; the difference of these two pairs of fractions are equal up to the sign! Therefore, the denominators of the differences of one pair are odd if and only if the denominators of the differences of the other pair are odd; so our relation $\sim$ is symmetric.

- **Transitivity**: take any $\left(\frac{a}{b}, \frac{c}{d}\right), \left(\frac{e}{f}, \frac{g}{h}\right), \left(\frac{i}{j}, \frac{k}{l}\right) \in \mathbb{Q}^2$, and suppose that $\left(\frac{a}{b}, \frac{c}{d}\right) \sim \left(\frac{e}{f}, \frac{g}{h}\right)$ and $\left(\frac{e}{f}, \frac{g}{h}\right) \sim \left(\frac{i}{j}, \frac{k}{l}\right)$. Then we have

$$\begin{aligned} \left(\frac{a}{b}, \frac{c}{d}\right) - \left(\frac{e}{f}, \frac{g}{h}\right) &= \left(\frac{\boxed{?}_1}{odd_1}, \frac{\boxed{?}_2}{odd_2}\right), \text{ and} \\ \left(\frac{e}{f}, \frac{g}{h}\right) - \left(\frac{i}{j}, \frac{k}{l}\right) &= \left(\frac{\boxed{?}_3}{odd_3}, \frac{\boxed{?}_4}{odd_4}\right). \end{aligned}$$

  Adding these two equations together gives us

$$\begin{aligned} \left(\frac{a}{b}, \frac{c}{d}\right) - \left(\frac{e}{f}, \frac{g}{h}\right) + \left(\frac{e}{f}, \frac{g}{h}\right) - \left(\frac{i}{j}, \frac{k}{l}\right) &= \left(\frac{\boxed{?}_1}{odd_1}, \frac{\boxed{?}_2}{odd_2}\right) + \left(\frac{\boxed{?}_3}{odd_3}, \frac{\boxed{?}_4}{odd_4}\right). \\ \Rightarrow \qquad \left(\frac{a}{b}, \frac{c}{d}\right) - \left(\frac{i}{j}, \frac{k}{l}\right) &= \left(\frac{\boxed{?}_1 \cdot \boxed{?}_2}{odd_1 \cdot odd_3}, \frac{\boxed{?}_3 \cdot \boxed{?}_4}{odd_2 \cdot odd_4}\right). \end{aligned}$$

  In other words: we have that the difference of $\left(\frac{a}{b}, \frac{c}{d}\right), \left(\frac{i}{j}, \frac{k}{l}\right)$ consists of a pair of fractions with odd denominators! This gives us $\left(\frac{a}{b}, \frac{c}{d}\right) \sim \left(\frac{i}{j}, \frac{k}{l}\right)$, and therefore that our relation is transitive.

The reason we care about this equivalence relation is the following observation:

**Observation.** If two points in $\mathbb{Q}^2$ are distance one apart, then they are equivalent under the equivalence relation $\sim$.

*Proof.* On the HW! $\square$

This observation is very useful for our goals: when we're coloring $\mathbb{Q}^2$, we will never have any edges connecting different equivalence classes! In other words, if we can simply create a coloring of each equivalence class of $\mathbb{Q}^2$ without conflicts, then we have a coloring of **all** of $\mathbb{Q}^2$ with no conflicts!

So: now, make the following second observation:

**Observation.** Let $E$ denote the equivalence class containing everything equivalent to $(0,0)$ under the relation $\sim$. Take any other point $(p_1, q_1) \in \mathbb{Q}^2$. Then the set

$$E_{(p_1,q_1)} = \{(e_1, e_2) + (p_1, q_1) \mid (e_1, e_2) \in E\}$$

is actually the equivalence class containing everything equivalent to $(p_1, q_1)$ under the relation $\sim$. In other words, every equivalence class under our relation is just a copy of $E$, translated by some constant.

*Proof.* This is not very difficult. First, take any two points $(e_1, e_2)+(p_1, q_1), (e_3, e_4)+(p_1, q_1)$ in $E_{(p_1,q_1)}$. Notice that because $(e_1, e_2)$ and $(e_3, e_4)$ are in the same equivalence class (and are thus equivalent to each other!), we have

$$(e_1, e_2) + (p_1, q_1) - ((e_3, e_4) + (p_1, q_1)) = (e_1, e_2) - (e_3, e_4) = \left(\frac{\boxed{?}_1}{odd_1}, \frac{\boxed{?}_2}{odd_2}\right).$$

This demonstrates that any two points in $E_{(p_1,q_1)}$ are equivalent. Now, take any point $(s_1, t_1)$ equivalent to $(p_1, q_1)$. Notice that because

$$\begin{aligned}(0,0) - ((s_1, t_1) - (p_1, q_1)) &= ((p_1, q_1) - (p_1, q_1)) - ((s_1, t_1) - (p_1, q_1))\\ &= (p_1, q_1) - (s_1, t_1)\\ &= \left(\frac{\boxed{?}_1}{odd_1}, \frac{\boxed{?}_2}{odd_2}\right),\end{aligned}$$

we can express $(s_1, t_1)$ as an element that is equivalent to $(0,0)$ plus $(p_1, q_1)$. This demonstrates that every element that is equivalent to $(p_1, q_1)$ lies in the set $E_{(p_1,q_1)}$.

By combining these two results, we have show that $E_{(p_1,q_1)}$ is the equivalence class corresponding to $(p_1, q_1)$, as desired. $\square$

Why is this nice? Well, it means that if we can color just **one** equivalence class — say, $E$, the equivalence class corresponding to $(0,0)$ — then we can color **every** equivalence class by just taking our coloring of $E$ and translating it by a constant!

So: if we can color $E$ without conflicts, then by translating this coloring to all of the other equivalence classes we have colored all of $\mathbb{Q}^2$ without conflicts! Therefore, it suffices to simply describe how to color $E$:

**Observation.** Take the set $E$ consisting of all of the points in $\mathbb{Q}^2$ equivalent to $(0,0)$ under the equivalence relation $\sim$. Color all points of the form $\left(\frac{odd_1}{odd_2}, \frac{odd_3}{odd_4}\right)$ and $\left(\frac{even_1}{odd_2}, \frac{even_3}{odd_4}\right)$ red, and all points of the form $\left(\frac{odd_1}{odd_2}, \frac{even_3}{odd_4}\right)$ and $\left(\frac{even_1}{odd_2}, \frac{odd_3}{odd_4}\right)$ blue.

Then no two points in $E$ that are distance 1 apart are colored the same color.

*Proof.* On the HW! $\square$

This concludes our proof: we have created a 2-coloring of the rational plane! Which isn't quite the same as coloring the real plane. However, many of the techniques that we've developed here will be useful to us in later talks and attempts to solve this problem! Progress. $\square$