Math/CCS 103 | Professor: Padraic Bartlett

Homework 8: Voting Systems

Due Monday, week 5 | *UCSB 2014*

For this problem set, we are working with the following conventions:

- $\mathbb{A}$ is some collection $\{A, B, C, \ldots\}$ of at least three options that we want voters to rank.
- For each i, we let R_i denote a **vote**, i.e. a **ranking** of our options. For example, if $\mathbb{A} = \{A, B, C\}$, then one ranking could be $A > B > C$.
- We denote a collection of n votes $(R_1, \ldots R_n)$ via the collection $\vec{R}$. Each coördinate of this "vector" is a ranking: i.e. $\vec{R}$ denotes things like

$$((A > B > C), (C > A > B), (C > B > A), (A > B > C))$$

- We denote the collection of all possible rankings on $\mathbb{A}$ via the symbol $\mathcal{R}$.
- Finally, we will denote a voting system c as just some function that takes in n votes and outputs some ranking. We formally write this as a function $c : \mathcal{R}^n \to \mathcal{R}$.
- With this set up, we will often look at what $c(\vec{R})$ is for some collection of votes $\vec{R}$: this is the "output" of the voting system c given the input collection of votes $\vec{R}$.
- In particular, we will consider a **fair** voting system c, as defined in class, and study its properties!

Prove at least **three** of the **four** claims below.

1. Take any choice A. Prove that there is some value i_A such that
   - if the first $i_A - 1$ voters $R_1, \ldots R_{i_A - 1}$ all place A at the bottom of their rankings, and
   - the last $n - i_A - 1$ voters $R_{i_A+1}, \ldots R_n$ all place A at the top of their rankings,

   then the i_A-th voter "gets to decide" whether society places A at the top or bottom of its rankings, in the following way: if R_{i_A} has A at the top of its rankings, then so does $c(\vec{R})$, and if R_{i_A} has A at the bottom of its rankings, then so does $c(\vec{R})$.

2. Take any option A. By using the independence of irrelevant alternatives property, extend problem 1 as follows: suppose that $\vec{R}$ is a collection of votes such that
   - the first $i_A - 1$ voters $R_1, \ldots R_{i_A - 1}$ all place A at the bottom of their rankings, and
   - the last $n - i_A - 1$ voters $R_{i_A+1}, \ldots R_n$ all place A at the top of their rankings.

   Then the i_A-th voter actually gets to decide the rankings of many other options, in this sense: if R_{i_A} ranks $C > A > B$ for two options $B, C \neq A$, then $C > A > B$ in $c(\vec{R})$.

3. By applying the irrelevance of independent alternative condition **again** to the result of problem 2, conclude the following: in **any** ranking $\vec{R}$, if C, B are any two non-A options and $C > B$ in R_{i_A}, then $C > B$ in $\vec{R}$. In this sense, R_{i_A} is a "dictator" for all non-A choices.

4. Take any two choices $A \neq B$, and find i_A, i_B for those choices. Prove that $i_A = i_B$. Conclude that R_{i_A} is a dictator, and that our voting scheme is a dictatorship.