The Euler Characteristic

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Definitions - Pronunciations!

Euler: "Oiler"

Planar: "Plainer"

Chi, χ: "Kai"

Definitions - Notation

For any graph, G, we say that G has:

- V vertices
- E edges
- F faces

Note that the infinite space around a graph counts as an additional face.

Definitions - Planar Graph

A Planar Graph is any 2D graph that has edges that do not cross.

Definitions - Formula

Mathematician Leonard Euler found that, for finite, connected, planar graphs,

$$V - E + F = 2$$

We call 2 the Euler Characteristic χ of a planar graph. In general:

$$\chi = V - E + F$$

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Proof by induction:
Take a graph with one vertex:

This graph has no edges, so it is trivially connected and planar. In this graph,

$$\chi = V - E + F = (1) - (0) + (1) = 2$$

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- Draw an edge from our vertex to itself: $(F + 1)$

In either situation, the additions even out in our formula.

$$\left.\begin{array}{l} \chi_1 = (V+1) - (E+1) + F \\ \chi_2 = V - (E+1) + (F+1) \end{array}\right\} = V - E + F = 2$$

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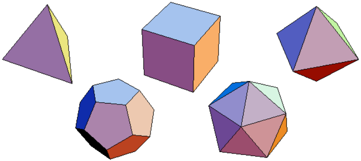
Absolutely nothing! In fact, we can create any planar graph with this method.

Imagine some arbitrarily large planar graph. If we add to this graph by one of our two methods, then we still have $\chi = 2$. Through induction, we have showed that *any* planar graph must have $\chi = 2$.

Proof for Polyhedra

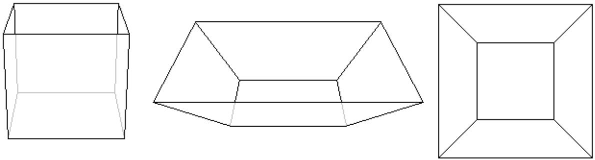
Euler's findings are not limited to 2D graphs. It is proven that $\chi = 2$ for convex, simple[1] polyhedra as well.



[1] Simple polyhedra have no holes in them, i.e. a torus is not simple.

Proof for Polyhedra

Cauchy's Proof: Take a polyhedron. Remove one of its faces. Looking at this empty face, "pull" the graph apart, creating a planar graph corresponding to the polyhedron. Since the resultant graph is planar, it must have $\chi = 2$! An example is shown below for a cube.



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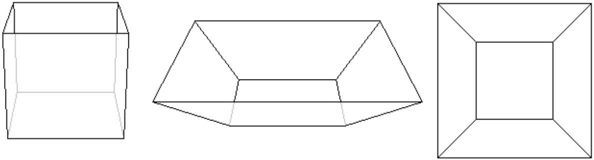

As it turns out, we don't care that we removed a face because it corresponds to the empty space around the planar graph.

Insight into other shapes

Remember the triangulation of a torus we used in class? We can use this triangulation to find the Euler characteristic of that torus! Take $H_{3,3}$.

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$V = 9$, $E = 27$, and $F = 18$, so $\chi = (12) - (36) + (24) = 0$

Why we care!

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For example, all of our simple, convex polyhedra are homeomorphic to a sphere. Imagine "inflating" them until they are round. Therefore, since χ is a topological invariant and they all have $\chi = 2$, a sphere has $\chi = 2$ as well.

Homework Questions

(Eric R.) A Soccer ball is made out of pentagons and hexagons that are stitched together. Given that a sphere has a Euler characteristic of 2, derive the number of hexagons and pentagons on the ball.

(Eric R.) *Expansion of the Utilities Problem:*

- There are three neighbors, and all of them want to connect to gas, water, and electricity. Is there a way for them all to do so such that no connection crosses another? Prove this is not possible.
- Find a shape on which it is possible, and explain why it works in terms of the Euler characteristic.

Sources:

- http://plus.maths.org/content/eulers-polyhedron-formula
- http://en.wikipedia.org/wiki/Euler_characteristic
- http://www.math.cornell.edu/~riley/Teaching/Topology2009/essays/Kalyanswamy.pdf
- http://www.math.rutgers.edu/~erowland/images/platonicsolids.gif
- http://mathworld.wolfram.com/Topology.html
- http://d2gbom735ivs5c.cloudfront.net/m/geometry/images/sphere-icosa.jpg
- http://www.mathsisfun.com/geometry/eulers-formula.html