Math/CS 103	Professor: Padraic Bartlett

Lecture 9: Understanding Null Space

Week 4	*UCSB 2013*

In our last full lecture, we introduced the concept of **null space**. In today's talk, we return to this object, with a new focus on **why** we care about the concept of the null space.

1 Null Space: A Motivating Example

First, a reminder of the definition we care about:

Definition. Pick two vector spaces V, W. Let $T : V \to W$ be a linear map from V to W.

The **null space** of T is the following set:

$$\text{null}(T) = \{\vec{v} \mid T(\vec{v}) = \vec{0} \in V\}$$

In other words, the null space of a linear map is the collection of all of the elements in V that T maps to 0.

In our last lecture, we said that the motivation for studying the **range** of a linear map was not too difficult to understand: given a linear map T, we often would want to understand what sorts of objects can be created as outputs of that map.

The null space was odder. On one hand, understanding the collection of all things that goes to 0 seems somewhat silly; why do we care so much about 0? Why not any other value?

To understand this, consider the map

$$T : \mathbb{R}^3 \to \mathbb{R},$$

defined by

$$T(x, y, z) = x + y.$$

The null space of this map is just the collection of all triples (x, y, z) such that

$$T(x, y, z) = 0;$$

i.e. it's the set

$$N(T) = \{(x, -x, z) : x, z \in \mathbb{R}\}.$$

So: here's a related question. What does the set of all vectors that map to 1 look like? Well, if we directly solve, we're looking for all triples (x, y, z) such that

$$T(x, y, z) = 1;$$

i.e. it's the set

$$\{(1+x, -x, z) : x, z \in \mathbb{R}\}.$$

In other words, it's basically what happens if we take $N(T)$ and scale every element in it by $(1, 0, 0)$!

Furthermore, if we take **any** real number $a \in \mathbb{R}$. we can see that

$$T(x, y, z) = a$$

if and only if our triple has the form

$$(a+x, -x, z),$$

for some $x, z \in \mathbb{R}$.

So, in a sense, when we understood the null space of the linear map T above, for any a we understood the collection of **all** elements that map to that a! So there's nothing special about 0, in a sense — rather, the null space appears to be capturing the total "redundancy" of our map, i.e. the number of elements that our maps sends to **any** element!

We make this rigorous with the following theorem:

Theorem 1. *Let $T : U \to V$ be a linear map. Let $N(T)$ denote the null space of T, and $\vec{u}, \vec{w}$ be any pair of vectors from U, V respectively such that $T(\vec{u}) = \vec{v}$.*

Let $T^{-1}(\vec{v})$ denote the set of all vectors in U that get mapped to $\vec{v}$ by T: i.e.

$$A_{\vec{v}} = \{\vec{w} \in U \mid T(\vec{w}) = \vec{v}\}.$$

Then $T^{-1}(\vec{v})$ is just $N(T)$ translated by $\vec{u}$! In other words,

$$T^{-1}(\vec{v}) = \{\vec{w} \in U \mid \textit{there is some } \vec{x} \in N(T) \textit{ such that } \vec{w} = \vec{x} + \vec{u}\}$$

In other words, understanding the collection of elements that all get mapped to $\vec{0}$ basically lets us understand the collection of elements that get mapped to any fixed vector $\vec{v}$.

Proof. Let $\vec{u}, \vec{w}$ be any pair of vectors from U, V respectively such that $T(\vec{u}) = \vec{v}$.

Take any vector $\vec{w} \in T^{-1}(\vec{v})$. By definition, we know that $T(\vec{w}) = \vec{v}$.

Look at the vector $\vec{w} - \vec{u}$. If we use the fact that T is linear, we can see that

$$T(\vec{w} - \vec{u}) = T(\vec{w}) - T(\vec{u}) = \vec{v} - \vec{v} = \vec{0};$$

therefore, $\vec{w} - \vec{u}$ is in the null space $N(T)$ of T. Therefore, we can write

$$\vec{w} = (\vec{w} - \vec{u}) + \vec{u};$$

i.e. we can write $\vec{w}$ as the sum of an element from $N(T)$ and the vector $\vec{u}$.

Now, take any vector $\vec{x} \in N(T)$. Again, because T is linear, we have

$$T(\vec{x} + \vec{u}) = T(\vec{x}) + T(\vec{u}) = \vec{0} + \vec{v} = \vec{v};$$

therefore, $\vec{x} + \vec{u}$ is in $T^{-1}(\vec{v})$.

So we've shown both that any element in $T^{-1}(\vec{v})$ can be written as the sum of $\vec{u}$ with an element of the null space of T, and furthermore that any such sum is an element of $T^{-1}(\vec{v})$. Therefore, these two sets are equal! □

People sometimes call these $T^{-1}(\vec{v})$ sets the "fibers" of the linear map T.

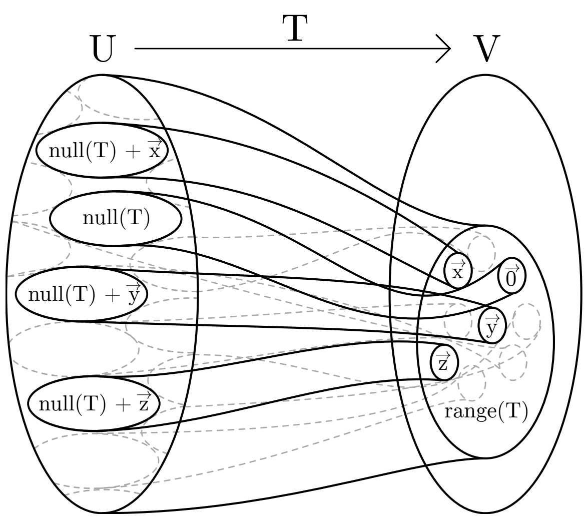
This theorem, hopefully, gives us some idea why we care about the null space: if we understand $T^{-1}(\vec{0})$, then we actually understand $T^{-1}(\vec{a})$, for **any** vector $\vec{a}$! That's powerful, and surprising.

But wait, there's more! Not only does this tell us what these $T^{-1}(\vec{a})$ things look like, it actually tells us what the entirety of U looks like in terms of the null space! Specifically, make the following two observations:

- Take any $\vec{u}$ in U. There is some set $T^{-1}(\vec{v})$ such that $\vec{u} \in T^{-1}(\vec{v})$. Specifically, just look at $T(\vec{u})$: this is equal to some element $\vec{a}$ in V. Then $\vec{u} \in T^{-1}(\vec{a})$, by definition.
- No vector $\vec{u}$ is in two different sets $T^{-1}(\vec{v}), T^{-1}(\vec{w})$. This is because if we apply T to any element in $T^{-1}(\vec{v})$, we get $\vec{v}$ by definition; similarly, if we apply T to any vector in $T^{-1}(\vec{w})$, we get $\vec{w}$ by definition. Therefore, if we had an element $\vec{u}$ in both sets, applying T to $\vec{u}$ would have to yield $\vec{v}$ and $\vec{w}$ simultaneously, which is only possible if $\vec{v} = \vec{w}$.

So the sets $T^{-1}(\vec{a})$ "partition" the set U: i.e. we can divide U up into various copies of these $T^{-1}(\vec{v})$ sets, such that every element of U is in exactly one of these sets! In other words, if we have a linear map $T : U \to V$, we can "chop up" U into a bunch of translated copies of the null space of T.

The diagram below may help you visualize this:



To make this diagram more concrete, consider the following example:

Example. Consider the linear map $T : \mathbb{R}^2 \to \mathbb{R}$, defined by $T(x, y) = 2x - y$. What is the null space of this map? What do the sets $T^{-1}(a)$ look like, for various values of $a \in \mathbb{R}$?

Answer. The null space of this map, by definition, is the set

$$\text{null}(T) = \{(x, y) \mid T(x, y) = 0\}.$$

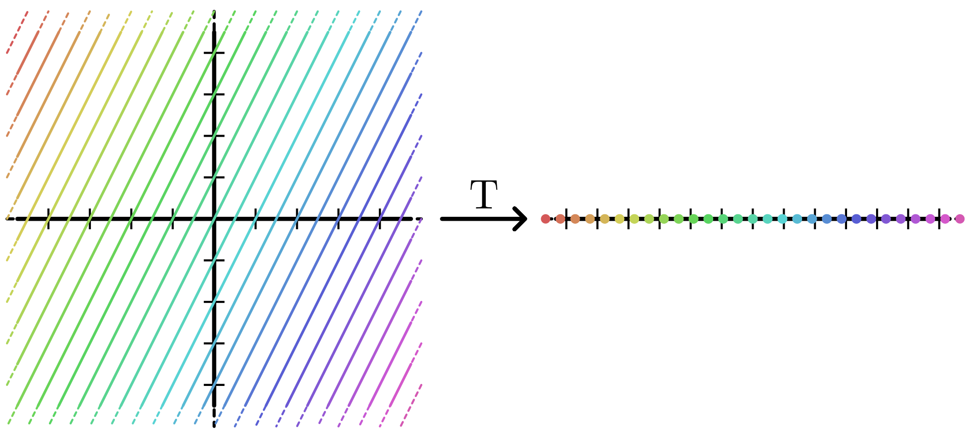
We know that $T(x,y) = 0$ if and only if $2x - y = 0$; in other words, whenever $2x = y$. Therefore, the null space of T can be more succinctly described as the set

$$\text{null}(T) = \{(x, 2x) \mid x \in \mathbb{R}\}.$$

Furthermore, notice that for any $a \in \mathbb{R}$, we have $T(a,0) = a$. Therefore, our theorem above tells us that we can express $T^{-1}(a)$ as the null space of T shifted by $(a,0)$: i.e.

$$T^{-1}(a) = \{(a + x, 2x) \mid x \in \mathbb{R}\}$$

Consequently, we can "partition" U into these $T^{-1}(a)$-sets, all of which are lines with slope 2 through the point $(a,0)$; each of these sets is then mapped to their corresponding value a by T. This can be visualized by the rather beautiful picture below:



Before we started this talk, we already understood why we cared about the range of a linear map T — it let us talk about the "outputs" of T. In a sense, the aim of this talk has been to show that understanding the null space of a linear map T performs a similar task: it gives us a ton of information about the "inputs" of T.