Math/CS 103	Professor: Padraic Bartlett

Lecture 8: Linear Transformations

Week 3	*UCSB 2013*

Today's lecture is on the concept of **linear maps**, which we've been studying in the past few problem sets. We start by reviewing the content from problem sets 6 and 7, and then move on to some new areas of study.

1 Linear Maps: Basics

On HW#6, we studied the concept of a **linear map,** defined as follows:

Definition. A **linear map** from a vector space V to another vector space W, where V and W may be different, is a function $T : V \to W$ with the following properties:

- **Plays well with addition**: for any $\vec{v}, \vec{w} \in V$, $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$.
- **Plays well with multiplication**: for any $\vec{v} \in V$ and any $a \in \mathbb{R}$, $T(a\vec{v}) = aT(\vec{v})$.

In the homework, we studied a number of maps, and showed that some of these were linear while others were not. We list a few of these proofs here, for completeness:

Claim 1. *Consider $T : \mathbb{R} \to \mathbb{R}$, defined such that*

$$T(x) = |x|.$$

This is not a linear map.

Proof. **This map does not play well with addition**. Specifically, notice that $T(1 + (-1)) = T(0) = 0$, while $T(1)+T(-1) = |1|+|-1| = 2$. These two quantities are different; therefore, our map is not additive. □

Claim 2. *Consider $T : \mathbb{R}^4 \to \mathbb{R}^2$, defined such that*

$$T(w, x, y, z) = (0, 0).$$

This is a linear map.

Proof. We simply show that it satisfies the following two properties:

- **It plays well with addition**. Take any two vectors $\vec{v}, \vec{w} \in \mathbb{R}^4$. We know that $T(\vec{v}+\vec{w}) = (0, 0)$, by definition. Similarly, we know that $T(\vec{v})+T(\vec{w}) = (0, 0)+(0, 0) = (0, 0)$. Therefore, these two quantities are the same for any two vectors $\vec{v}, \vec{w}$; therefore, our map is additive.
- **It plays well with multiplication**. Take any vector $\vec{v} \in \mathbb{R}^4$ and any $\lambda \in \mathbb{R}$. On one hand, we know that $T(\lambda\vec{v}) = (0, 0)$ by definition; on the other, we know that $\lambda T(\vec{v}) = \lambda(0, 0) = (0, 0)$. Therefore, this map is homogenous.

□

Claim 3. *Consider $T : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}$, defined such that*

$$T(p(x)) = p(1).$$

This is a linear map.

Proof. We show that it satisfies the following two properties:

- **It plays well with addition**. Take any two polynomials $p(x), q(x) \in \mathcal{P}_3(\mathbb{R})$. We know that $T(p(x)+q(x))$ is just the polynomial $p(x)+q(x)$ evaluated at $x = 1$, which is just $p(1) + q(1)$. On the other hand, $T(p(x)) + T(q(x)) = p(1) + q(1)$. These are equal; therefore, our map is additive.

- **It plays well with multiplication**. Take any polynomial $p(x) \in \mathcal{P}_3(\mathbb{R})$, and $\lambda \in \mathbb{R}$. We know that $T(\lambda p(x)) = \lambda \cdot p(1)$, and that $\lambda T(p(x)) = \lambda p(1)$. These two quantities are equal; therefore, this map is homogenous.

□

From here, the problem set handed out on Monday studied a pair of concepts related to the idea of **linear maps**, namely the idea of **subspace**, along with the concepts of **range** and **null space** of a linear map! We define these objects here:

2 Range and Null Space.

Definition. Let V be a vector space, like $\mathbb{R}^4$ or $\mathcal{P}_2(\mathbb{R})$. We say that some collection of vectors S from V is a **subspace** of V if it satisfies the following three properties:

- **Plays well with addition**. Given any two vectors $\vec{v}, \vec{w} \in S$, the sum $\vec{v} + \vec{w}$ is also contained in S.

- **Plays well with scalar multiplication**. Given any vector $\vec{v}$ and any real number $a \in \mathbb{R}$, the vector $a\vec{v}$ is also contained in S.

- **Not stupid**. S contains something: i.e. S is not the empty set $\emptyset$.

For example, you've shown on a previous problem set (HW#4, problem 4) that

$$S = \{p(x) \in \mathcal{P}_2(\mathbb{R}) : p(2) = 0\}$$

is a subspace of $\mathcal{P}_2(\mathbb{R})$. In particular, we did this by noticing that this subset

- **plays well with addition**. Given any two polynomials $p(x), q(x)$, if $p(2) = 0 = q(2)$, then $p(2) + q(2) = 0 + 0 = 0$. Therefore, $p(x) + q(x)$ is also contained in S.

- **plays well with scalar multiplication**. Given any polynomial $p(x)$ and any real number $a \in \mathbb{R}$, if $p(2) = 0$, then $a \cdot p(2) = a \cdot 0 = 0$, Therefore, $ap(x)$ is also contained in S.

- **isn't stupid**. S contains many elements, like (for example) $p(x) = x + 2$.

Similarly, on HW#3 problem 2(a), you showed that the set

$$R = \{(x, y, z)|x + y + z = 1\}$$

is **not** a subspace of $\mathbb{R}^3$. In particular, you noticed that it was possible to combine elements of R to get things outside of R itself: in particular, you guys found combinations of elements in R that could get any element in all of $\mathbb{R}^3$! For example,

$$(1, 0, 0) + (0, 1, 0) = (1, 1, 0),$$

which demonstrates that a sum of elements in R may not necessarily lie in R. Therefore, R is not a subspace.

In this problem set, we're going to study the following two objects:

Definition. Pick two vector spaces V, W. Let $T : V \to W$ be a linear map from V to W.

The **image** of T is the following set:

$$\text{im}(T) = \{T(\vec{v}) \mid \vec{v} \in V\}$$

In other words, the image of a linear map is the collection of all possible outputs of T under all possible inputs from V. Some people call this the **range** of T, and denote this range(T). Others will denote this $T(V)$, the idea being that you've put "all" of V into T itself.

Definition. Pick two vector spaces V, W. Let $T : V \to W$ be a linear map from V to W.

The **null space** of T is the following set:

$$\text{null}(T) = \{\vec{v} \mid T(\vec{v}) = \vec{0} \in V\}$$

In other words, the null space of a linear map is the collection of all of the elements in V that T maps to 0.

For example, consider the second linear map from HW#6: $T : \mathbb{R}^4 \to \mathbb{R}^2$,

$$T(w, x, y, z) = (0, 0).$$

For this map,

- The **image** of T is the set $\{(0, 0)\}$, because T outputs $(0, 0)$ on every input.
- The **null space** of T is all of $\mathbb{R}^4$, because T sends every element of $\mathbb{R}^4$ to $(0, 0)$.

Similarly, consider the map $T : \mathbb{R}^4 \to \mathbb{R}$, defined such that

$$T(x, y, z) = x + y + z.$$

Thing you should do if you don't believe it: show this is a linear map. Once you've done this, then you can easily check the following:

- The **image** of T is all of $\mathbb{R}$. This is because on input $(a, 0, 0)$, for any real number a, T outputs $a + 0 + 0 = a$. Therefore, we can get any real number as an output of T. Because T's output is restricted to $\mathbb{R}$, there's nothing else to worry about getting; consequently, the image of T is precisely T.
- The **null space** of T is the collection of all triples (a, b, c) such that $T(a, b, c) = a + b + c = 0$. In other words, if we solve for c in terms of the other two variables, it's the collection $\{(a, b, -a - b) : a, b \in \mathbb{R}\}$ of vectors in $\mathbb{R}^3$.

Something you will prove on the homework is the following:

- The **image** of a linear map $T : U \to V$ is a subspace of V.
- The **null space** of a linear map $T : U \to V$ is a subspace of U.

What I want to talk about here, briefly, is why we care about these objects. The motivation for linear maps is perhaps not too hard to understand: whenever we have any collection of objects in mathematics, one of the most natural things to study is functions between these objects. If we want these maps to preserve the structure that we care about with vector spaces — i.e. if we want our map to preserve the operations of vector addition and scalar multiplication, because those are the only two things we can do with vectors — we want this map to play nicely with addition and scalar multiplication, in pretty much precisely the way we ask linear maps to do.

The reason for studying the **range** is not much harder to understand. If we care about what some linear map $T : U \to V$ does, then we pretty clearly care about understanding the range of possible outputs that T has; this is one of the more fundamental things we can ask about T.

The **null space** is more obscure, but arguably the more important of the two. On one hand, understanding the collection of all things that goes to 0 seems somewhat silly; why do we care so much about 0? Why not any other value?

The reason for this comes from understanding how linear maps play with the null space.

For example, consider the map

$$T : \mathbb{R}^3 \to \mathbb{R},$$

defined by

$$T(x, y, z) = x + y.$$

The null space of this map is just the collection of all triples (x, y, z) such that

$$T(x, y, z) = 0;$$

i.e. it's the set

$$N(T) = \{(x, -x, z) : x, z \in \mathbb{R}\}.$$

So: here's a related question. What does the set of all vectors that map to 1 look like?

Well, if we directly solve, we're looking for all triples (x, y, z) such that

$$T(x, y, z) = 1;$$

i.e. it's the set

$$\{(1 + x, -x, z) : x, z \in \mathbb{R}\}.$$

In other words, it's basically what happens if we take $N(T)$ and scale every element in it by $(1, 0, 0)$!

Furthermore, if we take **any** real number $a \in \mathbb{R}$. we can see that

$$T(x, y, z) = a$$

if and only if our triple has the form

$$(a + x, -x, z),$$

for some $x, z \in \mathbb{R}$.

So, in a sense, when we understood the null space of the linear map T above, for any a we understood the collection of **all** elements that map to that a! So there's nothing special about 0, in a sense — rather, the null space appears to be capturing the total "redundancy" of our map, i.e. the number of elements that our maps sends to **any** element!

We make this rigorous with the following theorem:

Theorem 4. *Let $T : U \to V$ be a linear map. Let $N(T)$ denote the null space of T, and $\vec{u}, \vec{w}$ be any pair of vectors from U, V respectively such that $T(\vec{u}) = \vec{v}$.*

Let $A_{\vec{v}}$ denote the set

$$A_{\vec{v}} = \{\vec{w} \in U \mid T(\vec{w}) = \vec{v}\}.$$

Then $A_{\vec{v}}$ is just $N(T)$ translated by $\vec{u}$: i.e.

$$A_{\vec{v}} = \{\vec{w} \in U \mid \text{there is some } \vec{x} \in N(T) \text{ such that } \vec{w} = \vec{x} + \vec{u}\}$$

In other words, understanding the collection of elements that all get mapped to $\vec{0}$ basically lets us understand the collection of elements that get mapped to any fixed vector $\vec{v}$.

Proof. Let $\vec{u}, \vec{w}$ be any pair of vectors from U, V respectively such that $T(\vec{u}) = \vec{v}$.

Take any vector $\vec{w} \in A_{\vec{v}}$. By definition, we know that $T(\vec{w}) = \vec{v}$.

Look at the vector $\vec{w} - \vec{u}$. If we use the fact that T is linear, we can see that

$$T(\vec{w} - \vec{u}) = T(\vec{w}) - T(\vec{u}) = \vec{v} - \vec{v} = \vec{0};$$

therefore, $\vec{w} - \vec{u}$ is in the null space $N(T)$ of T. Therefore, we can write

$$\vec{w} = (\vec{w} - \vec{u}) + \vec{u};$$

i.e. we can write $\vec{w}$ as the sum of an element from $N(T)$ and the vector $\vec{u}$.

Now, take any vector $\vec{x} \in N(T)$. Again, because T is linear, we have

$$T(\vec{x} + \vec{u}) = T(\vec{x}) + T(\vec{u}) = \vec{0} + \vec{v} = \vec{v};$$

therefore, $\vec{x} + \vec{u}$ is in $A_{\vec{v}}$.

So we've shown both that any element in $A_{\vec{v}}$ can be written as the sum of $\vec{u}$ with an element of the null space of T, and furthermore that any such sum is an element of $A_{\vec{v}}$. Therefore, these two sets are equal! □

People sometimes call these $A_{\vec{v}}$ sets the "fibers" of the linear map T.

A similar theorem to the above is the following:

Definition. A map $T : U \to V$ is called an **injection** if no element in V is mapped to by more than one element.

Theorem 5. *A linear map $T : U \to V$ is an injection if and only if its null space consists of the single element $\vec{0}$.*

Proof. Suppose that a linear map T has two different $\vec{x} \neq \vec{y}$ such that $T(\vec{x}) = T(\vec{y})$.

Then

$$T(\vec{x} - \vec{y}) = T(\vec{x}) - T(\vec{y}) = \vec{0};$$

therefore, $\vec{x} - \vec{y}$ is in the null space of T. Moreover, this element is nonzero, because $\vec{x} \neq \vec{y}$.

Consequently, we have that whenever T is not an injection, T has some element $\vec{x} - \vec{y}$ that is in the null space that is nonzero.

This is the first half of our proof; we've shown that whenever T is not an injection, then there is something nontrivial in the null space.

To do the other half, we need to show that whenever T is an injection, then there is nothing other than 0 in the null space.

Well: on one hand, because T is linear, we know that T must map 0 to 0, because

$$T(\vec{0}) = T(\vec{0} + \vec{0}) = T(\vec{0}) + T(\vec{0}) = 2T(\vec{0}),$$

and the only vector that is equal to twice itself is the zero vector.

So we know that $\vec{0}$ is in T.

As well, because T is an injection, we know that no value in V is mapped to by more than one element. Therefore, because $\vec{0}$ is mapped to $\vec{0}$ and T is an injection, no other element can also map to $\vec{0}$; i.e. the null space of T is precisely the one element $\vec{0}$. □