Math/CS 103	Professor: Padraic Bartlett
Lecture 3: Linear Dependence and Independence	
Week 1	*UCSB 2013*

Hello! This mini-lecture is designed to introduce the concepts of **linear independence** and **dependence**. We do this below:

1 Linear Independence and Dependence

Most of the questions on the last problem set centered around the idea of a "tour:" i.e. a set of instructions that we can give a submarine's engines such that after performing all of these instructions, the submarine has returned to where it started. These conditions were coupled with the restriction that each engine is used only once.

Abstractly, then, we were asking the following question about vectors:

> "Given a collection $\{\vec{v_1}, \ldots \vec{v_n}\}$ of vectors in $\mathbb{R}^n$, when can we find constants $a_1, \ldots a_n \in \mathbb{R}$, not all zero, such that
>
> $$a_1\vec{v_1} + a_2\vec{v_2} + \ldots + a_n\vec{v_n} = \vec{0}$$
>
> holds?"

This is a frequently-studied question in mathematics, and it accordingly has a name!

Definition. Given a collection $\{\vec{v_1}, \ldots \vec{v_n}\}$ of vectors in $\mathbb{R}^n$, we say that this collection is **linearly dependent** if there are constants $a_1, \ldots a_n \in \mathbb{R}$, not all identically equal to zero, such that

$$a_1\vec{v_1} + a_2\vec{v_2} + \ldots + a_n\vec{v_n} = \vec{0}$$

holds.

If such constants do not exist, then we say that this collection is **linearly independent.**

To illustrate how this definition works, we study a few quick examples:

Question 1. *Is the collection* $\{(1,2,3),(4,5,6),(7,8,9)\}$ *linearly dependent?*

Answer. Yes! Notice that

$$(4,5,6) - (1,2,3) = (3,3,3),$$

and that

$$(1,2,3) + 2(3,3,3) = (7,8,9).$$

Therefore, we have

$$(1,2,3) + 2\left((4,5,6) - (1,2,3)\right) = (7,8,9),$$

which becomes (after subtracting $(7, 8, 9)$ from both sides)

$$(-1)(1, 2, 3) + 2(4, 5, 6) + (-1)(7, 8, 9) = (0, 0, 0).$$

Therefore, we have found a linear combination of vectors from our collection that sums to zero. This is the definition of linear dependence, as stated above.

Question 2. *Is the collection* $\{(1, 1, 0), (0, 1, 1), (1, 1, 1)\}$ *linearly dependent?*

Answer. No! To see why, consider any numbers a, b, c such that

$$a(1, 1, 0) + b(0, 1, 1) + c(1, 1, 1) = (0, 0, 0).$$

If we consider just the x-coördinate above, we can see that we must have

$$a \cdot 1 + b \cdot 0 + c \cdot 1 = 0,$$

i.e.

$$a = -c.$$

Similarly, if we consider the z-coördinate, we get that

$$a \cdot 0 + b \cdot 1 + c \cdot 1 = 0,$$

i.e.

$$b = -c.$$

Therefore, we must have that $a = b = -c$.

Now, if we consider the y-coördinate, we can also see that

$$a \cdot 1 + b \cdot 1 + c \cdot 1 = 0.$$

If we apply our identities $a = b = -c$, and replace a, b with $-c$ each, we get

$$-c + (-c) + c = 0,$$

i.e. $c = 0$, which forces a and b to also be 0.

In other words, we've just shown that if

$$a(1, 1, 0) + b(0, 1, 1) + c(1, 1, 1) = (0, 0, 0),$$

then a, b and c must all be identically 0. In other words, it is impossible for us to find constants a, b, c that are not identically 0 that satisfy the linear relation we're trying to show! Therefore, these vectors are linearly independent, as claimed.