Math/CS 103 Professor: Padraic Bartlett

Lecture 10: Injection, Surjection, and Linear Maps

Week 5 *UCSB 2013*

This talk is designed to go over some of the concepts we've been exploring recently with injections, surjections, and linear maps.

1 Injection, Surjection, Bijection and Size

We've been dealing with injective and surjective maps for a while now. Something you might have noticed, when looking at injective and surjective maps on finite sets, is the following triple of observations:

Observation. If $f : A \to B$ is an injective function and A, B are finite sets , then size$(A) \leq$ size(B).

The reasoning for this, in the finite case, is relatively simple:

1. If f is injective, then each element in A is being sent to a different element in B.
2. Thus, you'll need B to have at least size(A)-many elements, in order to provide that many targets.

Observation. If $f : A \to B$ is an surjective function and A, B are finite sets , then $|A| \geq |B|$.

Basically, this holds true because

1. Thinking about f as a collection of arrows from A to B, it has precisely $|A|$-many arrows by definition, as each element in A gets to go to precisely one place in B.
2. Thus, if we have to hit every element in B, and we start with only $|A|$-many arrows, we need to have $|A| \geq |B|$ in order to hit everything.

Observation. If $f : A \to B$ is an bijective function and A, B are finite sets, then $|A| = |B|$.

Proof. A bijection is a map that is both injective and surjective. If f is injective, then we know from our earlier work that $|A| \leq |B|$. If f is surjective, then we also know from our earlier work that $|A| \geq |B|$. Therefore, if we combine these observations, we have $|A| \leq |B|$ and $|A| \geq |B|$. The only way this is possible is if these two sets are the same size: i.e. if $|A| = |B|$. □

We can use this to come up with a notion of "size" that we can apply to all sets:

Definition. We say that two sets A, B are the same size (formally, we say that they are of the same **cardinality**,) and write $|A| = |B|$, if and only if there is a bijection $f : A \to B$.

This notion gives us some pretty paradoxical results! Consider the following pair of theorems:

2 Sizes of Infinity

Define the set

$$\mathbb{N} \cup \{\text{lemur}\} = \{a \mid \text{either } a \in \mathbb{N}, \text{ or } a = \text{lemur.}\}$$

This set is basically the same set as the natural numbers $\mathbb{N}$, except we've thrown in the element "lemur" as well[1]. This raises the following question:

Question. Are the sets $\mathbb{N}$ and $\mathbb{N} \cup \{\text{lemur}\}$ the same size?

Answer. We know that these two sets can be the same size if and only if there is a bijection between one and the other. So: let's try to make a bijection! In the typed notes, the suspense is somewhat gone, but (at home) imagine yourself taking a piece of paper, and writing out the first few elements of $\mathbb{N}$ on one side and of $\mathbb{N} \cup \{\text{lemur}\}$ on the other side. After some experimentation, you might eventually find yourself with the following map:

$$\begin{array}{ccc}
\boxed{\mathbb{N}} & & \boxed{\mathbb{N} \cup \{\text{lemur}\}} \\
1 & \longrightarrow & \text{lemur} \\
2 & \longrightarrow & 1 \\
3 & \longrightarrow & 2 \\
4 & \longrightarrow & 3 \\
5 & \longrightarrow & 4 \\
6 & \longrightarrow & 5 \\
\vdots & & \vdots
\end{array}$$

i.e. the map which sends 1 to the lemur and sends $n \to n-1$, for all $n \geq 2$. This is a bijection, because no element is mapped to twice and every element is mapped to at least once. Therefore, these sets are the same size!

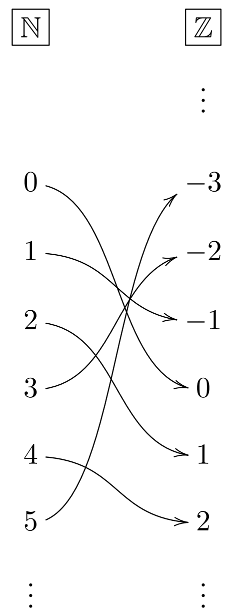
In a rather crude way, we have shown that adding one more element to a set as "infinitely large" as the natural numbers doesn't do anything to it! – the extra element just gets lost amongst all of the others. In other words, think of our bijection map as a way of "relabeling" elements: it takes any element n in the set $\mathbb{N}$ and sends it to (i.e. "relabels it as") some element in $\mathbb{N} \cup \{\text{lemur}\}$. What we've done here is shown that after relabeling, we can't tell these sets apart! — i.e. that in some sort of fundamental sense, these two sets are the same "size" in a way that two finite sets of different sizes cannot be.

This trick worked for one additional element. Can it work for infinitely many? Consider the next proposition:

Proposition. The sets $\mathbb{N}$ and $\mathbb{Z}$ are the same cardinality.

Proof. Consider the following map:

[1] In general, given a pair of sets A, B, we can form their **union**, denoted $A \cup B$. This set $A \cup B$ is the set consisting of all of the elements that were in either A or B: i.e. $A \cup B = \{x \mid \text{either } x \in A, \text{ or } x \in B, \text{ or possibly both.}\}$. For example, $\{1, 2\} \cup \{2, 5, \gamma\} = \{1, 2, 5, \gamma\}$.



i.e. the map which sends $n \to -(n-1)/2$ if n is odd, and $n \to n/2$ if n is even. This, again, is a bijection: the odd numbers $1, 3, 5, 7, 9, 11, \ldots$ get relabeled as the positive numbers $1, 2, 3, 4, 5, 6 \ldots$, and the even numbers $0, 2, 4, 6, 8, 10 \ldots$ get relabeled as the nonpositive numbers $0, -1, -2, -3, -4, -5 \ldots$. Therefore, these sets are the same cardinality. $\square$

There are many bijections we can create, and many sets we can show are the "same" size; you do this on the HW for several pairs of sets! With the rest of this talk, however, we switch back to looking at what we can do with linear algebra and these concepts.

3 Injection, Surjection and Linear Maps

Let $T : U \to V$ be a linear map between two vector space U, V. In this situation, the two concepts of **injection** and **surjection** are intimately related to the ideas of **null space** and **range** that we've been studying over the last week. One example is the following result, which we studied on the last pset:

- A map T is a surjection if and only if $\text{range}(T) = V$. This is literally from the definition of range, as the only way that $\text{range}(T)$ can equal V is if every element of V can arise as some output of T: i.e. if T is surjective.
- Slightly less obviously, a map T is an injection if and only if $\text{null}(T) = \vec{0}$. This comes from the theorem we proved in our last class, where we showed that whenever $T(\vec{x}) = \vec{y}$, then the sets $T^{-1}(\vec{y})$ have the form $\{\vec{x} + \vec{n} \mid \vec{n} \in \text{null}(T)\}$. If $\text{null}(T)$ is the set containing the single element $\{\vec{0}\}$, then the only element in $T^{-1}(\vec{y})$ is $\vec{x} + \vec{0} = \vec{x}$. Therefore, given any element $\vec{y}$ in the codomain V, there is at most one element in the domain U that maps to $\vec{y}$, because there is at most one element in $T^{-1}(\vec{y})$!

Combining these gives us problem 2# from the last problem set:

Theorem. Let $T : U \to V$ be a linear map between two vector space U, V. T is an isomorphism if and only if the following two conditions hold:

- $\mathrm{range}(T) = V$.
- $\mathrm{null}(T) = \{\vec{0}\}$.

There are tons of links between the ideas in linear maps and the ideas given by injection and surjection; your HW goes through many of them! One more that didn't make it to the set is the following:

Question. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a injective linear map. Prove that T is an isomorphism.

Proof. If T is an injective linear map, then the only condition we need to prove T satisfies is surjectivity: i.e. that for any vector $\vec{y}$ in $\mathbb{R}^3$, there is a vector $\vec{x} \in \mathbb{R}^3$ such that $T(\vec{x}) = \vec{y}$.

We prove this as follows.

1. Take the basis $B = \{(1,0,0),(0,1,0),(0,0,1)\}$. Look at the set $T(B) = \{T(1,0,0), T(0,1,0), T(0,0,1)\}$. We claim that this set spans $\mathbb{R}^3$. Notice that if this is true, we are done with our proof: to map to any $\vec{y} \in \mathbb{R}^3$, we simply use this spanning property and T's linearity to write

   $$\begin{aligned}\vec{y} &= aT(1,0,0) + bT(0,1,0) + cT(0,0,1) = T\left(a(1,0,0) + b(0,1,0) + c(0,0,1)\right)\\ &= T(a,b,c),\end{aligned}$$

   which demonstrates that for any $\vec{y}$, we can find a vector (a,b,c) that maps to $\vec{y}$.

   Therefore we just need to show that this set $T(B)$ spans $\mathbb{R}^3$. In fact, we're going to prove something stronger: $T(B)$ is a basis for $\mathbb{R}^3$!

2. We start by proving that $T(B)$ is linearly independent. This is not hard: take any nontrivial linear combination of the elements in $T(B)$:

   $$\begin{aligned}&aT(1,0,0) + bT(0,1,0) + cT(0,0,1)\\ =&T\left(a(1,0,0) + b(0,1,0) + c(0,1,0)+\right)\\ =&T(a,b,c).\end{aligned}$$

   If this is equal to $\vec{0}$, our map cannot be injective, because $T(0,0,0)$ maps to $(0,0,0)$ (as we proved on HW#4!) and (a,b,c) is nonzero (because the a,b,c correspond to a nontrivial linear combination) by assumption. Therefore, this linear combination is nonzero! So our set is linearly independent.

3. We now claim that $T(B)$ spans $\mathbb{R}^3$. To see why, simply notice that we have a set of three linearly independent vectors in $\mathbb{R}^3$. Geometrically, any such triple must span a three-dimensional space: therefore, because they are contained within $\mathbb{R}^3$, they specifically span all of $\mathbb{R}^3$ itself!

   Therefore, we have proven that $T(B)$ spans $\mathbb{R}^3$. By the reasoning in (1), we have consequently shown that T is surjective, as desired.

$\square$