Math/CS 103 Professor: Padraic Bartlett

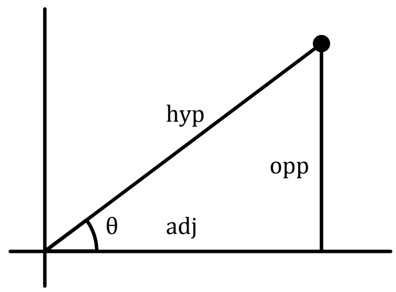
Homework 9: The Geometry of Linear Maps

Due 10/25/13, at the start of class. *UCSB 2013*

This problem set contains five questions. Choose **three** of the **five** problems below to complete by next class. Be ready and able to present your solutions if you have them, or your questions if you don't solve the problems!

Trigonometric tools you may need here:

- In the right triangle depicted below, we define $\cos(\theta)$ to be the fraction $\frac{\text{adj}}{\text{hyp}}$, and $\sin(\theta)$ to be the fraction $\frac{\text{opp}}{\text{hyp}}$.



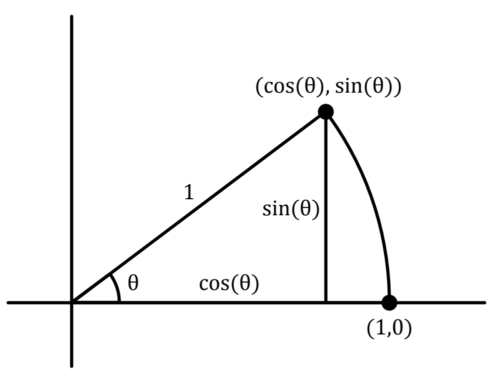
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.

1. Let $T : U \to V$ and $S : V \to W$ be a pair of linear maps. Prove that the map $S \circ T : U \to W$, defined by

$$S \circ T(\vec{u}) = S(T(\vec{u})),$$

is a linear map.

2. Consider the map $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$, that takes a vector (x, y) and rotates it by angle θ in a counterclockwise direction around the origin. For example, the vector $(1, 0)$ gets mapped to $(\cos(\theta), \sin(\theta))$, as depicted below:



(a) Show that the map T_θ is linear.

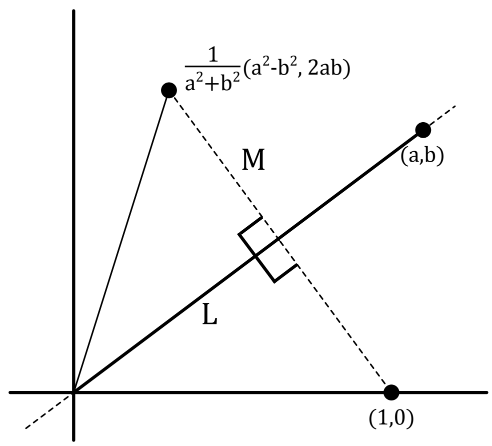
(b) Find coefficients $\alpha, \beta, \gamma, \delta$ such that

$$T(x, y) = (\alpha x + \beta y, \gamma x + \delta y).$$

3. Consider the map $R_{(a,b)} : \mathbb{R}^2 \to \mathbb{R}^2$, that takes a vector (x, y) and "reflects" it through the line spanned by the vector vector (a, b). Reflecting a point (x, y) through the line L spanned by a vector $\vec{v}$ is the following process:

   - Starting from (x, y), draw a line M perpindicular to L through the point (x, y).
   - Using this line, we can calculate the distance from our point (x, y) to the line L. Call this d.
   - Traveling towards the line L, start from (x, y) and move $2d$ units.
   - The point you arrive at is the reflection of (x, y) through the line L.

   For example, the point $(1, 0)$ gets mapped to $\frac{1}{a^2+b^2}\left(a^2 - b^2, 2ab\right)$ as depicted below:



   (a) Show that the map $R_{(a,b)}$ is linear.

   (b) Find coefficients $\alpha, \beta, \gamma, \delta$ such that

$$R_{(a,b)} = (\alpha x + \beta y, \gamma x + \delta y).$$

4. (a) Take any point $(a, b) \neq (0, 0)$. What is the map given by the composition $R_{(a,b)} \circ R_{(a,b)}$?

   (b) Take any angle θ. What is the map given by the composition $T_\theta \circ T_{-\theta}$?

5. Take any pair of reflections $R_{(a,b)}, R_{(c,d)}$. Let θ be the angle made by the ray connecting $(0, 0)$ to (a, b), and φ be the angle made by the ray connecting $(0, 0)$ to (c, d). Show that

$$R_{(c,d)} \circ R_{(a,b)} = T_{2\varphi - 2\theta}.$$