Math/CS 103 Professor: Padraic Bartlett

Homework + Lecture 7: Linear Map Properties

Due 10/18/13, at the start of class *UCSB 2013*

More definitions! In theory, we talked about all of these at the end of class on Monday.

1 Background.

Definition. Let V be a vector space, like $\mathbb{R}^4$ or $\mathcal{P}_2(\mathbb{R})$. We say that some collection of vectors S from V is a **subspace** of V if it satisfies the following three properties:

- **Plays well with addition**. Given any two vectors $\vec{v}, \vec{w} \in S$, the sum $\vec{v} + \vec{w}$ is also contained in S.
- **Plays well with scalar multiplication**. Given any vector $\vec{v}$ and any real number $a \in \mathbb{R}$, the vector $a\vec{v}$ is also contained in S.
- **Not stupid**. S contains something: i.e. S is not the empty set $\emptyset$.

For example, you've shown on a previous problem set (HW#4, problem 4) that

$$S = \{p(x) \in \mathcal{P}_2(\mathbb{R}) : p(2) = 0\}$$

is a subspace of $\mathcal{P}_2(\mathbb{R})$. In particular, we did this by noticing that this subset

- **plays well with addition**. Given any two polynomials $p(x), q(x)$, if $p(2) = 0 = q(2)$, then $p(2) + q(2) = 0 + 0 = 0$. Therefore, $p(x) + q(x)$ is also contained in S.
- **plays well with scalar multiplication**. Given any polynomial $p(x)$ and any real number $a \in \mathbb{R}$, if $p(2) = 0$, then $a \cdot p(2) = a \cdot 0 = 0$, Therefore, $ap(x)$ is also contained in S.
- **isn't stupid**. S contains many elements, like (for example) $p(x) = x + 2$.

Similarly, on HW#3 problem 2(a), you showed that the set

$$R = \{(x, y, z) | x + y + z = 1\}$$

is **not** a subspace of $\mathbb{R}^3$. In particular, you noticed that it was possible to combine elements of R to get things outside of R itself: in particular, you guys found combinations of elements in R that could get any element in all of $\mathbb{R}^3$! For example,

$$(1, 0, 0) + (0, 1, 0) = (1, 1, 0),$$

which demonstrates that a sum of elements in R may not necessarily lie in R. Therefore, R is not a subspace.

In this problem set, we're going to study the following two objects:

Definition. Pick two vector spaces V, W. Let $T : V \to W$ be a linear map from V to W. The **image** of T is the following set:

$$\text{im}(T) = \{T(\vec{v}) \mid \vec{v} \in V\}$$

In other words, the image of a linear map is the collection of all possible outputs of T under all possible inputs from V. Some people call this the **range** of T, and denote this range(T). Others will denote this $T(V)$, the idea being that you've put "all" of V into T itself.

Definition. Pick two vector spaces V, W. Let $T : V \to W$ be a linear map from V to W. The **null space** of T is the following set:

$$\text{null}(T) = \{\vec{v} \mid T(\vec{v}) = \vec{0} \in V\}$$

In other words, the null space of a linear map is the collection of all of the elements in V that T maps to 0.

For example, consider the second linear map from HW#6: $T : \mathbb{R}^4 \to \mathbb{R}^2$,

$$T(w, x, y, z) = (0, 0).$$

For this map,

- The **image** of T is the set $\{(0, 0)\}$, because T outputs $(0, 0)$ on every input.
- The **null space** of T is all of $\mathbb{R}^4$, because T sends every element of $\mathbb{R}^4$ to $(0, 0)$.

Similarly, consider the map $T : \mathbb{R}^4 \to \mathbb{R}$, defined such that

$$T(x, y, z) = x + y + z.$$

Thing you should do if you don't believe it: show this is a linear map. Once you've done this, then you can easily check the following:

- The **image** of T is all of $\mathbb{R}$. This is because on input $(a, 0, 0)$, for any real number a, T outputs $a + 0 + 0 = a$. Therefore, we can get any real number as an output of T. Because T's output is restricted to $\mathbb{R}$, there's nothing else to worry about getting; consequently, the image of T is precisely T.
- The **null space** of T is the collection of all triples (a, b, c) such that $T(a, b, c) = a + b + c = 0$. In other words, if we solve for c in terms of the other two variables, it's the collection $\{(a, b, -a - b) : a, b \in \mathbb{R}\}$ of vectors in $\mathbb{R}^3$.

2 Problems.

First, pick **one** of the **two** below to prove:

1. Show that for any linear map $T : V \to W$, the image of T is a subspace of W.
2. Show that for any linear map $T : V \to W$, the null space of T is a subspace of V.

Now, choose **four** of the **eight** maps below. For each map chosen, do the following:

- Calculate the **image** of the chosen map.
- Calculate the **null space** of the chosen map.
- Calculate the **dimension**[1] of both the image and the null space.

As always, show your work, and be ready to present your solutions in class!

3. $T : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}$, defined such that
$$T(p(x)) = p(3).$$

4. $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_5(\mathbb{R})$, defined such that
$$T(p(x)) = x^3 \cdot p(x).$$

5. $T : \mathbb{R}^n \to \mathbb{R}^{n-1}$, defined such that
$$T(x_1, \ldots x_n) = (x_2, x_3, \ldots x_n).$$

6. $T : \mathbb{R}^4 \to \mathbb{R}^4$, defined such that
$$T(w, x, y, z) = (w, w + x, w + x + y, w + x + y + z).$$

7. $T : \mathcal{P}_4(\mathbb{R}) \to T : \mathcal{P}_3(\mathbb{R})$, defined such that
$$T(p(x)) = \frac{d}{dx} p(x).$$

8. $T : \mathbb{R}^4 \to \mathbb{R}^2$, defined such that
$$T(w, x, y, z) = (w + x, y + z)\,.$$

9. $T : \mathcal{P}_1(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$, defined such that
$$T(p(x)) = (x - 3) \cdot p(x).$$

10. $T : \mathbb{R}^6 \to \mathbb{R}^6$, defined such that
$$T(u, v, w, x, y, z) = (z, y, x, w, v, u).$$

[1] If you've forgotten what dimension is, refer back to the third problem set / fourth set of lecture notes! In essence, however, the **dimension** of a given space is the number of elements in any basis for that space. For example, consider the linear map $T(x, y, z) = x+y+z$ we studied above. This space has image $\mathbb{R}$, which is one-dimensional because $\mathbb{R}$ has a basis with one element in it, namely $\{1\}$. Similarly, this space has null space $\{(a, b, -a - b) : a, b \in \mathbb{R}\}$. This null space has dimension 2, because we can find a basis for this set with two elements in it, namely $\{(1, 0, -1), (0, 1, -1)\}$. Because we can write any $(a, b, -a - b)$ as the sum $a(1, 0, -1) + b(0, 1, -1)$, and the only way for $a(1, 0, -1) + b(0, 1, -1) = (0, 0, 0)$ is if $a, b = 0$, this is a basis, and thus the dimension is 2.