Math/CS 103 — Professor: Padraic Bartlett

Homework 6: Linear Maps

Due 10/14/13, at the start of class — *UCSB 2013*

This set's a little different. First, recall from class that we've introduced two examples of finite-dimensional vector spaces:

- $\mathbb{R}^n$, the collection of all n-tuples of real numbers. For example, $\mathbb{R}^3$ was the collection of all triples (x, y, z) where x, y, z were all in $\mathbb{R}$.
- $\mathcal{P}_n(\mathbb{R})$, the collection of all polynomials with real-valued coefficients of degree $\leq n$. For example, $\mathcal{P}_2(\mathbb{R})$ is the collection of all polynomials of the form $a + bx + cx^2$, where a, b, c were in $\mathbb{R}$.

We called these both **vector spaces** because they were both objects that we had well-defined notions of **vector addition** (i.e. within any one of these spaces, we could add any two elements from that space together) and **scalar multiplication** (i.e. we could take any of these objects and multiply it by a real number, and still have an object in our space.)

In this problem set, you're going to work with the following definition:

Definition. A **linear map** from a vector space V to another vector space W, where V and W may be different, is a function $T : V \to W$ with the following properties:

- **Plays well with addition**: for any $\vec{v}, \vec{w} \in V$, $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$.
- **Plays well with multiplication**: for any $\vec{v} \in V$ and any $a \in \mathbb{R}$, $T(a\vec{v}) = aT(\vec{v})$.

For example, the map $id : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$id(x, y) = (x, y)$$

is a linear map, because

- **it plays well with addition**: for any two vectors $(a, b), (c, d)$, we have $id((a, b) + (c, d)) = id(a + c, b + d) = (a + c, b + d)$. This is the same thing as $id(a, b) + id(c, d) = (a, b) + (c, d) = (a + c, b + d)$.
- **it plays well with multiplication**: for any vector (a, b) and any real number λ, we have $id(\lambda(a, b)) = id(\lambda a, \lambda b) = (\lambda a, \lambda b)$. This is the same thing as $\lambda\ id(a, b) = \lambda(a, b) = (\lambda a, \lambda b)$.

Conversely, the map $T : \mathcal{P}_1(\mathbb{R}) \to \mathcal{P}_1(\mathbb{R})$, defined by

$$T(a + bx) = a^2$$

is not a linear map, because

- **it does not play well with addition**. Specifically, look at the two polynomials $2, 2 + x$ in $\mathcal{P}_1(\mathbb{R})$. $T(2 + (2 + x)) = T(4 + x) = 4^2 = 16$, while $T(2) + T(2 + x) = 2^2 + 2^2 = 8$.

Choose **six** of the **twelve** maps below, and decide whether or not they are **linear maps**. If you choose something that is a linear map, show that it obeys the two properties described in the definition of a linear map. If you choose something that is not a linear map, construct a set of inputs that makes it fail one of the two properties described in the definition of a linear map. Have fun!

1. $T : \mathbb{R} \to \mathbb{R}$, defined such that
$$T(x) = |x|.$$

2. $T : \mathbb{R}^4 \to \mathbb{R}^2$, defined such that
$$T(w, x, y, z) = (0, 0).$$

3. $T : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}$, defined such that
$$T(p(x)) = p(1).$$

4. $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_5(\mathbb{R})$, defined such that
$$T(p(x)) = x^2 \cdot p(x).$$

5. $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_5(\mathbb{R})$, defined such that
$$T(p(x)) = (1 + x) \cdot p(x).$$

6. $T : \mathbb{R}^n \to \mathbb{R}^{n-1}$, defined such that
$$T(x_1, \ldots x_n) = (x_2, x_3, \ldots x_n).$$

7. $T : \mathbb{R}^3 \to \mathcal{P}_3(\mathbb{R})$, defined such that
$$T(a, b, c) = (x - a) \cdot (x - b) \cdot (x - c).$$

8. $T : T : \mathcal{P}_4(\mathbb{R}) \to T : \mathcal{P}_3(\mathbb{R})$, defined such that
$$T(p(x)) = \frac{d}{dx} p(x).$$

9. $T : \mathbb{R}^4 \to \mathbb{R}^2$, defined such that
$$T(w, x, y, z) = \left(2w + 3x, 4y + \left(5^{5^5}\right) z\right).$$

10. $T : \mathbb{R}^4 \to \mathbb{R}^2$, defined such that
$$T(w, x, y, z) = (wx, yz).$$

11. $T : \mathbb{R}^6 \to \mathbb{R}^6$, defined such that
$$T(u, v, w, x, y, z) = (z, y, x, w, v, u).$$

12. $T : \mathbb{R}^2 \to \mathbb{R}$, defined such that
$$T(x, y) = \left(x^3 + y^3\right)^{1/3}.$$

Bonus! Create a map $T : \mathbb{R} \to \mathbb{R}$ that satisfies the "plays well with addition" property, but fails the "plays well with multiplication" property. Or show it's impossible.