Math/CS 103 Professor: Padraic Bartlett

Homework 4: Calculators!

Due 10/7/13, at the start of class *UCSB 2013*

Choose **three** of the **five** problems below to complete by next class. Be ready and able to present your solutions if you have them, or your questions if you don't solve the problems!

1. Your submarine company from the past two problem sets has been bought out by Texas Instruments. Now, you make calculators.

   Your first job at this new company is the following: you've been put in charge of handling low-degree polynomial calculations. In particular, you've been put in charge of creating a device that can evaluate any degree-3 or lower polynomial.

   The trick is this: the only chips available to you are the random ones we had lying about in the back of the storeroom. Specifically, these were the five chips you found while rummaging around:

    : Chip 1: on input x, outputs $1 + x$.

    : Chip 2: on input x, outputs $x + x^2$.

    : Chip 3: on input x, outputs $x^2 + x^3$.

    : Chip 4: on input x, outputs $1 + x^3$.

    : Chip 5: on input x, outputs x.

   In software, you have the following ways to alter these chips:

   - Scaling: if we have a chip that on input x produces $p(x)$, we can **scale** this chip's output by any **nonzero** quantity α, so that it instead produces $\alpha p(x)$ For example, because we have a chip that on input x produces $1 + x$, we can **scale** this chip by a factor of 2 so that it instead produces $2 + 2x$.
   - Adding: if we have two (possibly scaled) chips, we can combine these chips in a way that sums their output. For example, we have a chip that on input x produces $1 + x$ and another chip that on input x produces $x + x^2$. Therefore, we can scale the first chip by 1, the second chip by -1, and combine their results to get a chip that on input x produces $1 - x^2$.

   Take any arbitrary degree-3 polynomial $a + bx + cx^2 + dx^3$. Given these five chips, can you combine them with the scaling and adding operations in software to create a device that on input x, outputs this polynomial? Describe an algorithm to perform this process if it is possible, or construct a counterexample if it is not possible. Recall that you cannot scale things by 0.

2. Oh noes! You've accidentally stepped on one of your chips. Can you still create your degree-3 polynomial calculator? For each chip, determine if you can still create any degree-3 polynomial, or if there are polynomials you cannot create.

3. You've used up all of your chips from problem 1. However, you find a large bucket of new chips! From looking at the label on the bucket and examining several chips, you have determined that for any polynomial $p(x) = a + bx + cx^2 + dx^3$ with the following properties:

   - a, b, c, d are all integers between -10 and 10, and
   - $p(1) = 1$,

   we have a chip that on input x returns that polynomial.

   Using these chips, can you still create your degree-3 polynomial calculator? Either choose appropriate chips and create an algorithm if one exists, or prove that it is impossible.

4. Oh noes (part 2): your bucket from question 3 has mysteriously caught fire! Being an electrical engineer is hard.

   Luckily, you've found a third bucket, full of yet more chips! These have the following properties: for any polynomial $p(x) = a + bx + cx^2 + dx^3$ with the following properties:

   - a, b, c, d are all integers between -10 and 10, and
   - $p(2) = 0$,

   we have a chip that on input x returns that polynomial.

   Using these chips, can you still create your degree-3 polynomial calculator? Either choose appropriate chips and create an algorithm if one exists, or prove that it is impossible.

5. New management has taken over your division! On the plus side: you now have a bunch of shiny new chips! In fact, for any polynomial $a + bx + cx^2 + dx^3$ where a, b, c, d are integers between -100 and 100, you have chips that on input x output that polynomial.

   On the down side: management is currently insisting that you create a device capable of calculating **any** degree-three polynomial, using **only** three chips. If you cannot, you will be fired.

   Prove that you are fired.