Math/CS 103 | Professor: Padraic Bartlett

Homework 18: How to Calculate the Determinant

Due 12/2/13, at the start of class. | *UCSB 2013*

On Wednesday, we defined the determinant (as opposed to the "positive determinant,") and showed that it was **n-linear**. On this problem set, we do a bunch of things with this definition. Do a total of **two** problems in this set!

1 Alternate Formulations of the Determinant

In class on Monday, many of you used other definitions of the determinant! This section is proving that these are all equivalent to the definition we've been using in class, that comes from volume.

Definition. A **permutation** of the set $\{1, \ldots n\}$ is any ordered way to write down the symbols $\{1, \ldots n\}$. For example, the collection of all permutations of the string $(1, 2, 3)$ is the set

$$(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1).$$

Given a permutation π, we refer to the k-th entry of π by writing $\pi(k)$. For example, if $\pi = (2, 3, 4, 1)$, we would interpret $\pi(2)$ to be the second entry of π, which is 3.

Theorem. Take any permutation. We claim that it can be created by the following process:

1. Start with the permutation $(1, 2, 3, \ldots n)$.
2. Repeatedly pick pairs of elements in the permutation we have, and swap them.
3. By carefully choosing the pairs in step 2 above, we can get to any other permutation.

The total number of swaps used above is called the **signature** of that permutation. For example, the permutation $(2, 3, 4, 1)$ has signature $\operatorname{sgn}((2, 3, 4, 1)) = 3$, because

$$(1, 2, 3, 4) \xrightarrow{\text{switch } 1,2} (2, 1, 3, 4) \xrightarrow{\text{switch } 1,3} (2, 3, 1, 4) \xrightarrow{\text{switch } 1,4} (2, 3, 4, 1).$$

A given permutation can have different signatures; for example, we could have written the above with five swaps, by simply taking the above set of three swaps and then switching 1 and 3 back and forth twice (as swapping two numbers twice doesn't change anything.) However, it is impossible for a permutation to have both an odd and an even signature! (You're asked to prove this on the HW, but you're welcome to use it without proof on other problems!)

In particular, this means that for any permutation π, the quantity

$$(-1)^{\operatorname{sgn}}(\pi)$$

is well-defined; i.e. it's 1 for any permutation that can be created with an even number of swaps, and -1 for a permutation that can be created with an odd number of swaps. We use this in the determinant definitions below!

1. Let A be a $n \times n$ matrix, of the form

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

Consider the following object:

$$\sum_{\substack{\text{all of the permutations} \\ \pi \text{ of the set } \{1,\ldots n\}}} (-1)^{\operatorname{sgn}(\pi)} \cdot a_{1,\pi(1)} \cdot a_{2,\pi(2)} \cdot \ldots \cdot a_{n,\pi(n)}.$$

For example, for 3×3 matrices, this is the sum

$$\begin{aligned} &(-1)^{\operatorname{sgn}((1,2,3))} \cdot a_{11}a_{22}a_{33} + (-1)^{\operatorname{sgn}((1,3,2))} \cdot a_{11}a_{23}a_{32} + (-1)^{\operatorname{sgn}((2,1,3))} \cdot a_{12}a_{21}a_{33} \\ &+(-1)^{\operatorname{sgn}((2,3,1))} \cdot a_{12}a_{23}a_{31} + (-1)^{\operatorname{sgn}((3,1,2))} \cdot a_{13}a_{21}a_{32} + (-1)^{\operatorname{sgn}((3,2,1))} \cdot a_{13}a_{22}a_{31}, \end{aligned}$$

which if you calculate the signatures is just

$$\begin{aligned} &(-1)^0 \cdot a_{11}a_{22}a_{33} + (-1)^1 \cdot a_{11}a_{23}a_{32} + (-1)^1 \cdot a_{12}a_{21}a_{33} \\ &+(-1)^2 \cdot a_{12}a_{23}a_{31} + (-1)^2 \cdot a_{13}a_{21}a_{32} + (-1)^1 \cdot a_{13}a_{22}a_{31}. \end{aligned}$$

Show that for 3×3 matrices, this object is the same thing as the determinant. (Hint: just use the fact that the determinant is n-linear!)

2. If you did problem 1, generalize your argument to $n \times n$ matrices!

3. Show that the signature, discussed above, is well-defined. In other words, show that no permutation can be written as both an odd number of swaps and an even number of swaps.

4. Problem 1 is one method for calculating the determinant, which some of of you used on the HW. Another method people used on the HW was the following:

   Let A be a $n \times n$ matrix. Given a row i and a column j, let A_{ij} denote the $n-1 \times n-1$ matrix formed by deleting the i-th row and j-th column of A. Consider the object formed by doing the following:

$$a_{11} \cdot \det(A_{11}) - a_{21} \cdot \det(A_{21}) + a_{31} \cdot \det(A_{31}) \ldots + (-1)^{n-1} a_{n1} \cdot \det(A_{n1}).$$

   For example, for a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, this is just

$$a_{11} \cdot \det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) - a_{21} \cdot \det\left(\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}\right) + a_{31} \cdot \det\left(\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}\right).$$

   Show that for a 3×3 matrix, this is the same thing as the determinant! (Again, use n-linearity.)

5. Generalize your proof for 4 to $n \times n$ matrices!

2 Putnam Problems!

1. Let A be a $n \times n$ matrix with entries $a_{ij} = |i - j|$. For example, when $n = 3$, we have

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

   For any n, find the determinant of A.

2. Let A be a 3×2 matrix and B be a 2×3 matrix, such that

$$AB = \begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{bmatrix}.$$

   Show that

$$BA = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

3. Consider the following 2-player game:

   - You start with a 10×10 matrix.
   - Player 1 and Player 2 alternate putting real numbers into the entries of the matrix.
   - Once the matrix is filled, player 1 wins if the determinant of this matrix is nonzero; player 2 wins if the determinant is zero.

   Which player has a winning strategy? I.e. if you were challenged to this game, would you prefer to go first or second? In either case, what would your strategy be?