Math/CS 103 — Professor: Padraic Bartlett

Homework 13: More Matrices!

Due 11/8/13, at the start of class. — *UCSB 2013*

Thing I realized over the weekend: if you don't know how to do problem 2 from HW#12, it's likely that the entire idea of compositions of linear maps will not make lots of sense, and this problem set will fail. So, this HW starts off with a proof of said problem! After that, it has two theoretical problems that build off of that, and four computational problems. You should do **three** questions in total, out of the **six** listed. Even if you don't do the two theory problems, you should read their statements and the section afterwards that describes how to use them, so that you can use their results in the computational section.

1 A Proof of Problem 2, HW#12

On the last homework, some of you (hopefully!) studied the following problem:

Problem. Take any pair of linear maps $A, B : \mathbb{R}^n \to \mathbb{R}^n$ with associated matrices

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix}, B = \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,n} \\ b_{2,1} & b_{2,2} & \dots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \dots & b_{n,n} \end{bmatrix}.$$

Let $\vec{e_1}$ denote the standard basis vector for $\mathbb{R}^n$ with a 1 in its 1st coördinate and 0's elsewhere. What is $B \circ A(\vec{e_1})$?

Answer. The answer is the following:

$$B \circ A(\vec{e_1}) = \left(\vec{a_1} \cdot \vec{b_1}, \vec{a_1} \cdot \vec{b_2}, \dots, \vec{a_1} \cdot \vec{b_n}\right).$$

In this section, we present a proof of this claim!

Proof. To find $B \circ A$ as applied to $\vec{e_1}$, we just apply A to $\vec{e_1}$, and then apply B to the result of that calculation.

By definition, $A(\vec{e_1})$ is just the first column of A: i.e.

$$(a_{1,1}, a_{2,1}, \dots a_{n,1}).$$

To calculate $B(a_1, a_2, \dots a_n)$, we just use the fact that B is linear to write

$$\begin{aligned} B(a_{1,1}, a_{2,1}, \dots a_{n,1}) &= B(a_{1,1}, 0 \dots 0) + B(0, a_{2,1}, 0 \dots 0) + \dots + B(0, \dots 0, a_{n,1}) \\ &= a_{1,1}B(\vec{e_1}) + a_{2,1}B(\vec{e_2}) + \dots + a_{n,1}B(\vec{e_n}). \end{aligned}$$

From here, we just use the definition of B to see that $B(\vec{e_i})$ is precisely the i-th column of B, and therefore that

$$\begin{aligned} B(a_{1,1}, a_{2,1} \dots a_{n,1}) &= a_{1,1}B(\vec{e_1}) + a_{2,1}B(\vec{e_2}) + \dots + a_{n,1}B(\vec{e_n}) \\ &= a_{1,1}(b_{1,1}, \dots b_{n,1}) + a_{2,1}(b_{1,2}, \dots b_{n,2}) + \dots + a_{n,1}(b_{1,n}, \dots, b_{n,n}). \end{aligned}$$

The first coördinate of this vector is precisely the sum

$$a_{1,1}b_{1,1} + a_{2,1}b_{1,2} + a_{3,1}b_{1,3} + \dots + a_{n,1}b_{1,n},$$

which we can recognize as the dot product $\vec{a_1} \cdot \vec{b_1}$. Similarly, the second coördinate of $B(a_{1,1}, a_{2,1}, \dots a_{n,1})$ is just

$$a_{1,1}b_{2,1} + a_{2,1}b_{2,2} + a_{3,1}b_{2,3} + \dots + a_{n,1}b_{2,n}.$$

Let $\vec{a_i}$ denote the i-th column of A: i.e. $\vec{a_i} = (a_{1,i}, a_{2,i}, \dots, a_{m,i})$. Similarly, let $\vec{b_j}$ denote the j-th row of B: i.e. $\vec{b_j} = (b_{j,1}, b_{j,2}, \dots, b_{j,m})$. If we do this, we can recognize the expression above as the dot product $\vec{a_1} \cdot \vec{b_2}$. By similar logic, we can see that the k-th coördinate of $B(a_{1,1}, a_{2,1}, \dots a_{n,1})$ is just the dot product of $\vec{a_1}$ and $\vec{b_k}$. Combining these observations together proves our claim:

$$B \circ A(\vec{e_1}) = \left(\vec{a_1} \cdot \vec{b_1}, \vec{a_1} \cdot \vec{b_2}, \dots, \vec{a_1} \cdot \vec{b_n}\right).$$

□

Yay, ok. Why do you care? Well, because it lets you understand generalizations!

2 Questions on matrix composition

1. Take any pair of linear maps $A : \mathbb{R}^n \to \mathbb{R}^m, B : \mathbb{R}^m \to \mathbb{R}^k$ with associated matrices

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}, B = \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,m} \\ b_{2,1} & b_{2,2} & \dots & b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k,1} & b_{k,2} & \dots & b_{k,m} \end{bmatrix}.$$

Let $\vec{e_i}$ denote the i-th standard basis vector for $\mathbb{R}^n$, with a 1 in its i-th coördinate and 0's elsewhere. Let $\vec{a_i}$ denote the i-th column of A: i.e. $\vec{a_i} = (a_{1,i}, a_{2,i}, \dots, a_{m,i})$. Similarly, let $\vec{b_j}$ denote the j-th row of B: i.e. $\vec{b_j} = (b_{j,1}, b_{j,2}, \dots, b_{j,m})$.

By slightly modifying the proof above, prove that $B \circ A$ applied to the standard basis vector $\vec{e_i} \in \mathbb{R}^n$ is just the vector

$$\left(\vec{a_i} \cdot \vec{b_1}, \vec{a_i} \cdot \vec{b_2}, \dots, \vec{a_i} \cdot \vec{b_k}\right).$$

2. Assume that you've proven 1 above, even if you haven't. By using the definition of how to turn linear maps into matrices, show that the matrix corresponding to their composition $B \circ A : \mathbb{R}^n \to \mathbb{R}^k$ is the $k \times n$ matrix

$$\begin{bmatrix} \vec{b_1} \cdot \vec{a_1} & \vec{b_1} \cdot \vec{a_2} & \dots & \vec{b_1} \cdot \vec{a_n} \\ \vec{b_2} \cdot \vec{a_1} & \vec{b_2} \cdot \vec{a_2} & \dots & \vec{b_2} \cdot \vec{a_n} \\ \dots & \dots & \ddots & \dots \\ \vec{b_k} \cdot \vec{a_1} & \vec{b_k} \cdot \vec{a_2} & \dots & \vec{b_k} \cdot \vec{a_n} \end{bmatrix}.$$

For those of you that have seen matrix multiplication before, **that's what we just did.** Basically, if you have two linear maps written as matrices, we can use the dot product as described in question 2 to compose them! Even if you don't solve problem 2, I highly recommend you remember the **result** from problem 2, as it is an extremely useful tool for composing matrices.

If you don't quite get what these results are saying, maybe the following example will help:

Example. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map

$$T = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix},$$

and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

What is the matrix given by the composition $S \circ (T \circ S)$?

Answer. If we just apply problem 2 from the above section, we have

$$\begin{aligned}(T \circ S) = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \cdot (0,1) & \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \cdot (1,0) \\ \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot (0,1) & \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot (1,0) \end{bmatrix} \\ &= \begin{bmatrix} \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}S \circ (T \circ S) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} &= \begin{bmatrix} (0,1) \cdot \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) & (0,1) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \\ (1,0) \cdot \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) & (1,0) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.\end{aligned}$$

In other words, this is the map that sends $(1,0)$ to $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and $(0,1)$ to $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. If you recall our discussion from the past homework set, this is in fact the rotation matrix that rotates $\mathbb{R}^2$ by $-\pi/4$!

We can double-check this answer by thinking geometrically: the map T is just the matrix given by the linear map $T_{\pi/4}$ that rotates space by $\pi/4$ radians, while the map S is the matrix that flips the x and y-coördinates. Composing these maps as $S \circ T \circ S$, geometrically speaking, should give you a map that first switches the x and y coordinates, then rotates by $\pi/4$ in the "switched" space, then flips back — which is just rotation by $-\pi/4$!

3 Questions on things that actually involve concrete matrices

1. In the last HW, we looked at the map $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$, that rotates a vector in the plane counterclockwise by theta. Specifically, we looked at this linear map, and showed that it is the matrix

$$T_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Take two such matrices T_θ, T_γ, and (using the matrix composition formula discussed above!) calculate the matrix

$$T_\theta \circ T_\gamma.$$

Show that it is equal to the matrix $T_{\gamma+\theta}$.

2. Find a linear map $E_{\lambda,i} : \mathbb{R}^n \to \mathbb{R}^n$ that multiplies its i-th coördinate by λ and does not change any of the others. Write this map as a matrix. For an arbitrary $n \times n$ matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix},$$

find the composition $E_{\lambda,i} \circ A$.

3. Find a linear map $E_{(i,j)} : \mathbb{R}^n \to \mathbb{R}^n$ that switches its i-th and j-th coördinates and does not change any of the others. Write this map as a matrix. For an arbitrary $n \times n$ matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix},$$

find the composition $E_{(i,j)} \circ A$.

4. Find a linear map $E_{\lambda i \mapsto j} : \mathbb{R}^n \to \mathbb{R}^n$ that adds λ copies of its i-th coördinate to its j-th coördinate and does not change any of the others. Write this map as a matrix. For an arbitrary $n \times n$ matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix},$$

find the composition $E_{\lambda i \mapsto j} \circ A$.

5. In HW #12, one linear map that we looked at was the identity map $id(\vec{x})$, that sends a vector $\vec{x}$ to itself: i.e. $id(\vec{x}) = \vec{x}$. We showed that this linear map corresponded to the following matrix, which we called the **identity matrix**:

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

For any $n \in \mathbb{N}$, find a matrix A such that A^n is the identity matrix, but A^k is not the identity matrix, for any $1 \leq k < n$.