Math/CS 103 Professor: Padraic Bartlett

Homework 12: Matrices

Due 11/4/13, at the start of class. *UCSB 2013*

Here's an observation some of you have occasionally made on the HW:

Observation. Take a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$. Let the vectors $\vec{e_1}, \ldots \vec{e_n}$ denote the standard basis vectors for $\mathbb{R}^n$: i.e. $\vec{e_1} = (1, 0, \ldots 0), \vec{e_2} = (0, 1, 0 \ldots 0), \ldots \vec{e_n} = (0, 0 \ldots 0, 1)$.

Then: if we know where T sends these basis vectors, then we know where T sends **any** vector in $\mathbb{R}^n$! Specifically: suppose we know the quantities

$$T(\vec{e_1}), T(\vec{e_2}), \ldots T(\vec{e_n}).$$

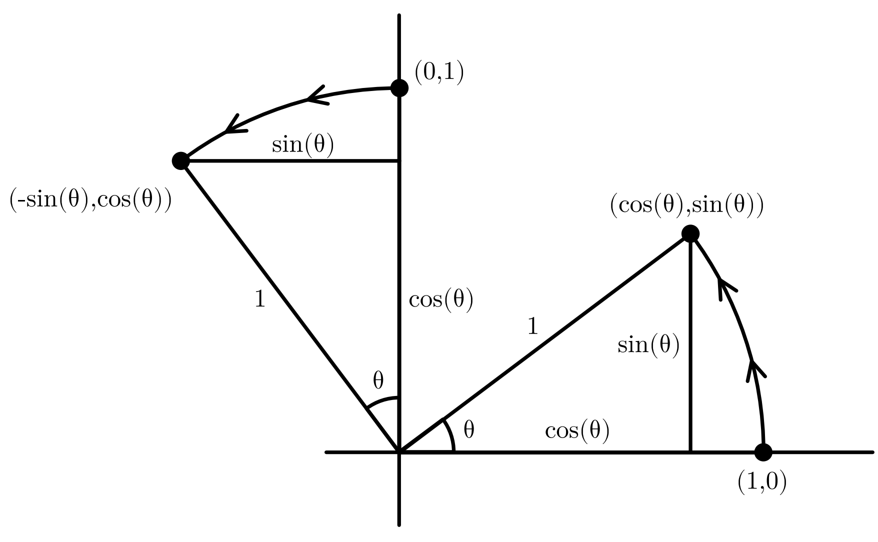
Take any vector $(x_1, \ldots x_n) \in \mathbb{R}^n$. Because T is linear, we then have that

$$\begin{aligned} T(x_1, \ldots x_n) &= T(x_1 \cdot \vec{e_1} + \ldots + x_n\vec{e_n}) \\ &= x_1 T(\vec{e_1}) + \ldots + x_n T(\vec{e_n}). \end{aligned}$$

In other words, we don't actually have to calculate $T(x_1, \ldots x_n)$! Instead, we can just use our prior knowledge of what the $T(\vec{e_i})$'s are to figure this out indirectly.

This can be remarkably useful: as you've seen before, some maps are really hard to understand in general, but very easy to evaluate on some set of basis vectors for $\mathbb{R}^n$.

For example, recall homework 9, where you studied the rotation map $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$. This map took a vector (x, y) and rotated it by angle θ in a counterclockwise direction around the origin. For example, the vector $(1, 0)$ was mapped to $(\cos(\theta), \sin(\theta))$, and the vector $(0, 1)$ would be mapped to $(-\sin(\theta), \cos(\theta))$, as the diagram below illustrates:



Using this observation, if we know that this map T_θ is linear, then we can simply write

$$\begin{aligned} T_\theta(x, y) = T_\theta(x(1, 0) + y(0, 1)) = xT_\theta(1, 0) + yT_\theta(0, 1) &= x(\cos(\theta), \sin(\theta)) + y(-\sin(\theta), \cos(\theta)) \\ &= (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta)). \end{aligned}$$

This is a much simpler and easier way to calculate $T_\theta(x, y)$ than the direct fashion (as many of you realized on said problem set :p) In fact, this technique is so valuable that we have developed notation and definitions to specifically work with this idea!

Definition. Take a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$. Let the vectors $\vec{e_1}, \ldots \vec{e_n}$ denote the standard basis vectors for $\mathbb{R}^n$: i.e. $\vec{e_1} = (1, 0, \ldots 0), \vec{e_2} = (0, 1, 0 \ldots 0), \ldots \vec{e_n} = (0, 0 \ldots 0, 1)$.

For each of the vectors $T(\vec{e_i})$ in $\mathbb{R}^m$, write

$$T(\vec{e_i}) = (t_{1,i}, t_{2,i}, \ldots, t_{m,i}),$$

where the values $t_{i,j}$ are all real numbers

We turn T into an $m \times n$ **matrix**, i.e. a $m \times n$ grid of real numbers, as follows:

$$T \longrightarrow T_{\text{matrix}} = \begin{bmatrix} t_{1,1} & t_{1,2} & \ldots & t_{1,n} \\ t_{2,1} & t_{2,2} & \ldots & t_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{m,1} & t_{m,2} & \ldots & t_{m,n} \end{bmatrix}.$$

In other words,

$$T \longrightarrow T_{\text{matrix}} = \begin{bmatrix} \vdots & \vdots & \ldots & \vdots \\ T(\vec{e_1}) & T(\vec{e_2}) & \ldots & T(\vec{e_n}) \\ \vdots & \vdots & \ldots & \vdots \end{bmatrix},$$

Similarly, given some $m \times n$ matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{bmatrix},$$

we can interpret A as a **linear map** $A_{\text{map}} : \mathbb{R}^n$ to $\mathbb{R}^m$ as follows:

- For any of the standard basis vectors $\vec{e_i}$, we define $A_{\text{map}}(\vec{e_i})$ to simply be the vector $(a_{1,i}, \ldots a_{m,i})$.
- For any other vector $(x_1, \ldots x_n) \in \mathbb{R}^n$, we define $A_{\text{map}}(x_1, \ldots x_n)$ to simply be the corrresponding linear combination of the $\vec{e_i}$'s: i.e.

$$A_{\text{map}} : (x_1, \ldots x_n) := x_1 \cdot A_{\text{map}}(\vec{e_1}) + \ldots + x_n A_{\text{map}}(\vec{e_n}).$$

In practice, we will usually not bother writing the subscripts "map" and "matrix" on these objects, and think of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $m \times n$ matrices as basically the same things.

For example, return to the map $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$. Because this map sends $(1, 0)$ to $(\cos(\theta), \sin(\theta))$, and $(0, 1)$ to $(-\sin(\theta), \cos(\theta))$, we would express this map as a matrix as follows:

$$T_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

This problem set is split into two sections: a handful of **theoretical** properties about matrices, and a handful of **concrete calculations** that ask you to find matrices associated to various linear maps. Complete **one** problem from the theoretical section, and **three** from the concrete section.

1 Theoretical properties of matrices

1. Take any linear map $A : \mathbb{R}^n \to \mathbb{R}^m$. Let A have the associated matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}.$$

   Denote the row vectors $(a_{i,1}, \dots a_{i,n}) \in \mathbb{R}^n$ of our matrix A with the vectors $\vec{a_i}$, for shorthand. Show that for any $\vec{x} \in \mathbb{R}^n$, we have

$$A(\vec{x}) = (\vec{x} \cdot \vec{a_1}, \vec{x} \cdot \vec{a_2}, \dots \vec{x} \cdot \vec{a_m})$$

2. Take any pair of linear maps $A, B : \mathbb{R}^n \to \mathbb{R}^n$ with associated matrices

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix}, B = \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,n} \\ b_{2,1} & b_{2,2} & \dots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \dots & b_{n,n} \end{bmatrix}.$$

   Calculate $B \circ A$ when it is applied to $\vec{e_1} = (1, 0, 0, \dots 0)$.

3. Let $A, B : \mathbb{R}^n \to \mathbb{R}^m$ be a pair of linear maps, with associated matrices

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}, B = \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,n} \\ b_{2,1} & b_{2,2} & \dots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \dots & b_{m,n} \end{bmatrix}.$$

   Define the linear map $(A + B) : \mathbb{R}^n \to \mathbb{R}^m$ as follows: for any $\vec{x} \in \mathbb{R}^n$, $(A + B)(\vec{x}) = A(\vec{x}) + B(\vec{x})$. Find the matrix corresponding to $(A + B)$.

2 Explicitly calculating matrices for various linear maps

There are several linear maps below. Pick three and write them as matrices.

1. $T : \mathbb{R}^6 \to \mathbb{R}^6$, defined such that

$$T(u, v, w, x, y, z) = (z, y, x, w, v, u).$$

2. $T : \mathbb{R}^n \to \mathbb{R}^{n-1}$, defined such that

$$T(x_1, \dots x_n) = (x_2, x_3, \dots x_n).$$

3. $S^{-1} \circ T \circ S : \mathbb{R}^4 \to \mathbb{R}^4$, where $S : \mathbb{R}^4 \to \mathcal{P}_3(\mathbb{R})$, $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ are the maps

$$S(a, b, c, d) = (a + bx + cx^2 + dx^3), \text{ and}$$
$$T\left(a + bx + cx^2 + dx^3\right) = \frac{d}{dx}\left(a + bx + cx^2 + dx^3\right).$$

4. $R \circ T \circ S : \mathbb{R}^4 \to \mathbb{R}^4$, where $S : \mathbb{R}^4 \to \mathcal{P}_3(\mathbb{R})$, $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R})$ and $R : \mathcal{P}_4(\mathbb{R}) \to \mathbb{R}^5$ are the maps

$$S(a, b, c, d) = (a + bx + cx^2 + dx^3),$$
$$T\left(a + bx + cx^2 + dx^3\right) = \int_0^x (a + bt + ct^2 + dt^3)dt, \text{ and}$$
$$R\left(a + bx + cx^2 + dx^3 + ex^4\right) = (a, b, c, d, e).$$

5. $T : \mathbb{R}^n \to \mathbb{R}^n$, defined such that

$$T(\vec{x}) = \vec{x}.$$

6. $T : \mathbb{R}^4 \to \mathbb{R}^4$, defined such that

$$T(w, x, y, z) = (w, w + x, w + x + y, w + x + y + z).$$