Math/CS 103	Professor: Padraic Bartlett

Homework 11: More Injections, Surjections and Linear Maps

Due 11/1/13, at the start of class. *UCSB 2013*

This problem set contains eight questions. Choose **three** of the **eight** problems below to complete by next class. Be ready and able to present your solutions if you have them, or your questions if you don't solve the problems!

1 Linear algebra-ish problems.

1. Suppose that S is a linear map $\mathbb{R}^3 \to \mathbb{R}$ with $\text{null}(S) = \{(x, x + y, y) : x, y \in \mathbb{R}\}$. Prove that S is surjective.

2. Let T be a linear map from $\mathbb{R}^2 \to \mathbb{R}^3$, such that $\text{range}(T) = \{(x, x + y, y) \mid x, y \in \mathbb{R}\}$. Show that T must be injective.

3. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a surjective linear map. Prove that T is an isomorphism.

4. (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map of the form $T(x, y) = (ax + by, cx + dy)$. Show that T is an injection if and only if $ad - bc \neq 0$.

   (b) How many of these maps exist if we limit a, b, c, d to integers between 1 and 50? (A general formula for a, b, c, d integers between 1 and n would also be interesting.)

2 Injection/surjection problems in general.

1. Is there a bijection between the set $[0, 1]$ and the set $(0, 1)$?

2. Is there a bijection between $(0, 1)$ and $\mathbb{R}$?

3. Given a set A, we say that a set B is a **subset** of A if every member of B is also a member of A. For example, if $A = \{w, x, y, z\}$ and $B = \{w, z\}$, then B is a subset of A, because every member of B is also a member of A. Conversely, the set $C = \{x, y, z, \alpha\}$ is not a subset of A, because the element α is a member of C and not a member of A.

   A special set that bears mentioning is the **empty set**, denoted $\emptyset$. This is the set that contains no elements.

   Given a set A, the **power set** of A, denoted $\mathcal{P}(A)$, is the collection of all of the subsets of A. For example, the power set of $\{1, 2, 3\}$ is the set containing the following eight sets:

   - $\emptyset$
   - $\{1\}$
   - $\{2\}$
   - $\{3\}$
   - $\{1, 2\}$
   - $\{1, 3\}$
   - $\{2, 3\}$
   - $\{1, 2, 3\}$

We write this as

$$\mathcal{P}(A) = \Bigg\{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \Bigg\}.$$

Show that for any set A, there is no injection from A to $\mathcal{P}(A)$. (In particular, make sure your proof works when A contains infinitely many elements.)

4. Is there an injection from $\mathcal{P}(\mathbb{N})$ to the real numbers $\mathbb{R}$? Either construct one if it exists, or prove it cannot exist.