Math/CS 103 Professor: Padraic Bartlett

Homework 10: Injection, Bijection, Surjection

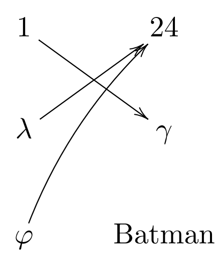
Due 10/28/13, at the start of class. *UCSB 2013*

To do this problem set, you need the following definitions:

Definition. A **function** f with domain A and codomain B, formally speaking, is a collection of pairs (a, b), with $a \in A$ and $b \in B$, such that there is exactly one pair (a, b) for every $a \in A$. Informally speaking, a function $f : A \to B$ is just a map which takes each element in A to an element in B.

Examples.

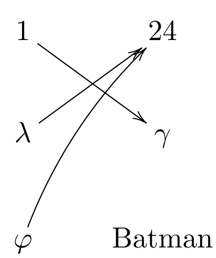
- $f : \mathbb{Z} \to \mathbb{N}$ given by $f(n) = 2|n| + 1$ is a function.
- $g : \mathbb{N} \to \mathbb{N}$ given by $g(n) = 2|n| + 1$ is also a function. It is in fact a different function than f, because it has a different domain!
- $j : \mathbb{N} \to \mathbb{N}$ defined by $h(n) = n^2$ is yet another function
- The function j depicted below by the three arrows is a function, with domain $\{1, \lambda, \varphi\}$ and codomain $\{24, \gamma, \text{ Batman}\}$:



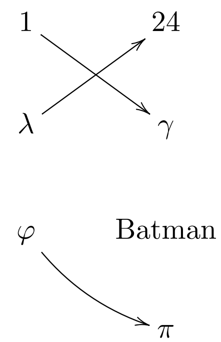
  It sends the element 1 to γ, and the elements λ, φ to 24. In other words, $h(1) = \gamma$, $h(\lambda) = 24$, and $h(\varphi) = 24$.

Definition. We call a function f **injective** if it never hits the same point twice – i.e. for every $b \in B$, there is **at most one** $a \in A$ such that $f(a) = b$.

Examples. The function h from before is not injective, as it sends both λ and φ to 24:



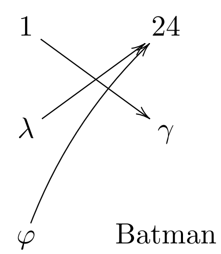
However, suppose that we add a new element π to our codomain, and make φ map to π. Then, this modified function is now injective, because no two elements in its domain are sent to the same place:



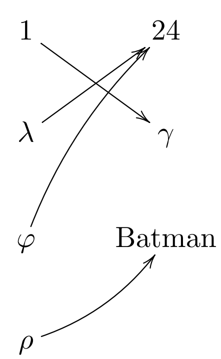
A converse concept to the idea of injectivity is that of **surjectivity**, as defined below:

Definition. We call a function f **surjective** if it hits every single point in its codomain – i.e. if for every $b \in B$, there is **at least one** $a \in A$ such that $f(a) = b$.

Examples. The function h from before is not injective, as it doesn't send anything to Batman:



However, if we add a new element ρ to our domain, and make ρ map to Batman, our function is now surjective, as it hits all of the elements in its codomain:



Definition. We call a function **bijective** if it is both injective and surjective.

This problem set contains six questions. Choose **three** of the **six** problems below to complete by next class. Be ready and able to present your solutions if you have them, or your questions if you don't solve the problems!

1. Is there a bijection between the set of all possible words and $\mathbb{N}$?

   Specifically: let's suppose that we're limiting ourselves to the 26-character Latin alphabet, and that the only kinds of things that can be **words** are finite strings of characters from the Latin alphabet. So things like

   - rabbit
   - barglearglesnarg
   - ssss
   - froyo

   are all possibly words. Call the set of all possible words $\mathbb{W}$. Is there a bijection between $\mathbb{W}$ and $\mathbb{N}$?

2. Let U, V be a pair of vector spaces. A map $T : U \to V$ is called an **isomorphism** if it is both a linear map and a bijection.

   Show that a linear map $T : U \to V$ is an isomorphism if and only if T satisfies the following two properties:

   - $\text{null}(T) = \{\vec{0}\}$.
   - $\text{range}(T) = V$.

3. Suppose that $T : U \to V$ is an isomorphism.

   (a) Show that $T^{-1}(\vec{v})$ is a set containing exactly one element, for any $\vec{v} \in V$.

   (b) Using part (a), define the map $T^{inverse} : V \to U$ as follows:

   $$T^{inverse}(\vec{v}) = \text{ the unique vector } \vec{u} \text{ in } T^{-1}(\vec{v}).$$

   Show that this[1] map $T^{inverse}$ is an isomorphism.

4. For each of the following pairs of vector spaces, either construct an isomorphism between those two vector spaces, or explain why no isomorphism exists.

   (a) $U = \mathbb{R}^3$, $V = \mathcal{P}_2(\mathbb{R})$.

   (b) $U = \{(x, y, z) : x + y + z = 0\}$, $V = \mathbb{R}^2$.

   (c) $U = \mathbb{R}^5$, $V = \{(x, y, x, y) \mid x, y \in \mathbb{R}\}$.

5. Is there a surjection from $\mathbb{N}$ to $\mathbb{N}^2 = \{(a, b) \mid a, b \in \mathbb{N}\}$? Either find such a map if it exists, or prove that no such map can exist.

6. Is there an injection from $\mathbb{R}^2 \to \mathbb{R}$? Either find one if it exists, or show that no such map can exist.

[1] In practice, most people will denote $T^{inverse}$ as T^{-1}, and therefore use the same object to both describe the sets $T{-}1(\vec{v})$ and the map $T^{inverse} : U \to V$. A thing to know when you read other literature.