Math 7h	Professor: Padraic Bartlett

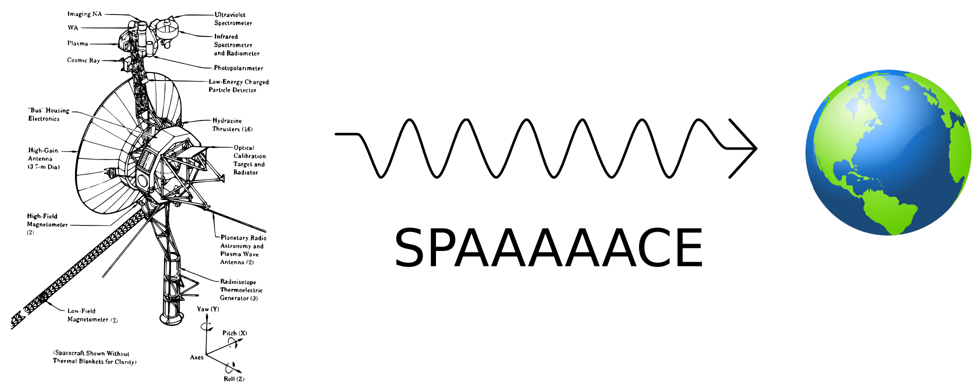
Lecture 8: Error-Correcting Codes and Latin Squares

Week 8 *UCSB 2014*

1 Error-Correcting Codes

Consider the following problem:

Problem. Suppose that you are the Voyager 1 probe. You are currently on the outer limits of the solar system, and about to leave the solar system forever! Consequently, you want to call your parents. However, you are currently separated from your parents by the vast interstellar void of space.


Imaging NA
WA
Plasma
Cosmic Ray
Ultraviolet Spectrometer
Infrared Spectrometer and Radiometer
Photopolarimeter
Low-Energy Charged Particle Detector
"Bus" Housing Electronics
Hydrazine Thrusters (16)
High-Gain Antenna (3.7-m Dia)
Optical Calibration Target and Radiator
High-Field Magnetometer (2)
Planetary Radio Astronomy and Plasma Wave Antenna (2)
Radioisotope Thermoelectric Generator (3)
Low-Field Magnetometer (2)
Yaw (Y)
Pitch (X)
Axes
Roll (Z)
(Spacecraft Shown Without Thermal Blankets for Clarity)
SPAAAAACE

The vast interstellar void of space has an annoying habit of occasionally containing stray electromagnetic waves that interfere with your communications back home; when you send a signal back home (in binary, naturally), occasionally one of your 1's will be switched into a 0, or vice-versa. Assume that no more than one out of three consecutive bits in any message you send will be scrambled.

How can you call home?

One solution to this problem you might come up with is to simply "build redundancy" into the signals you send home, by sending (say) six 1's every time you went to send a 1, and six 0's every time you went to send a 0. For example, to send the message "0101," you'd send

$$000000111111000000111111.$$

Then, even if some of the bits are flipped, your parents back on earth could still decode your message. In particular, if at most one bit out of any consecutive three is flipped, each of your all-0's blocks will have at most two 1's in them after errors are introduced, while each of your all-1's blocks will have at most two 0's in them after errors. In either case, we would never confuse one of these blocks with the other: if your parents received the signal

$$010010001111100001101011,$$

they'd break it up into the four blocks

010010001111100001101011,

and "correct" the errors to

000000111111000000111111,

which is unambiguously the signal 0101.

This code can correct for the presence of one error out of any three consecutive blocks, but no better (i.e. if we could have more than two errors in a block of six, we might have three errors in a string of six: in this case it would be impossibe to tell what our string was intended to be. For example, the string 000111 could have resulted from three errors on the signal 000000, or three errors on the signal 111111.) It can accomplish this at the cost of sending $6k$ bits whenever it wants to transmit k bits of information.

Can we do better? In specific, can we make a code that is more efficient (i.e. needs less bits to transmit the same information,) or can correct for more errors? With a little thought, it's easy to improve our code above: if we instead simply replace each 0 with just 000 and each 1 with 111, our code can still correct for the presence of at most one error in any three consecutive blocks (for example, 101 is unambiguously the result of one error to 111,) and now needs to send just $3k$ bits to transmit k bits of information.

There are more interesting codes than just these repetition codes: consider for example the codeword table

word	signal to transmit
000	000000
100	100011
010	010101
001	001110
011	011011
101	101101
110	110110
111	111000

In this code, we encode messages by breaking them into groups of three, and then replacing each string of three with the corresponding group of six. For example, the message "010 101 111" would become

010101101101111000.

In this code, every word in the table above differs from any other word in at least three spots (check this!) Therefore, if we have at most 1 error in any six consecutive bits, we would never confuse a word here with any other word: changing at most one bit in any block of six would still make it completely unambiguous what word we started with.

Therefore, if we sent the string that we described above, and people on Earth received

010111101111110000,

they would first break it into groups of six

$$010111101111110000,$$

and then look through our codeword table for what words these strings of six could possibly be, if at most one error in every six consecutive bits could occur:

$$010101101101111000.$$

This then decodes to "010 101 111," the message we sent.

This code can correct for at most one error in any six consecutive bits (worse than our earlier code,) but does so much more efficiently: it only needs to send $2k$ bits to transmit a signal with k bits of information in it.

So: suppose we know ahead of time the maximum number of errors in any consecutive string of symbols. What is the most efficient code we can make to transmit our signals?

At this time, it makes sense to try to formalize these notions of "maximum number of errors" and "efficiency." Here are a series of definitions, that formalize the words and ideas we've been playing with in this talk:

Definition. A q-ary code C of length n is a collection C of words of length n, written in base q. In other words, C is just a subset of $(\mathbb{Z}/q\mathbb{Z})^n$.

Example. The "repeat three times" code we described earlier is a 2-ary code of length 3, consisting of the two elements $\{(000), (111)\}$. We used it to encode a language with two symbols, specifically 0 and 1.

The second code we made is a 2-ary code of length 6, consisting of the 8 elements we wrote down in our table.

Definition. Given a q-ary code C of length n, we define its **information rate** as the quantity

$$\frac{\log_q(\text{\# of elements in C})}{n}$$

This, roughly speaking, captures the idea of how "efficient" a code is.

Example. The "repeat three times" code we described earlier contains two codewords of length 3; therefore, its information rate is

$$\frac{\log_2(2)}{3} = \frac{1}{3}.$$

This captures the idea that this code needed to transmit three bits to send any one bit of information.

Similarly, the second code we made contains 8 codewords of length six, and therefore has information rate

$$\frac{\log_2(8)}{6} = \frac{3}{6} = \frac{1}{2}.$$

Again, this captures the idea that this code needed to transmit two bits in order to send any one bit of information.

Definition. The **Hamming distance** $d_H(\mathbf{x}, \mathbf{y})$ between any two elements $\mathbf{x}, \mathbf{y}$ of $(\mathbb{Z}/q\mathbb{Z})^n$ is simply the number of places where these two elements disagree.

Given a code C, we say that the minimum distance of C, $d(C)$, is the smallest possible value of $d_H(\mathbf{x}, \mathbf{y})$ taken over all distinct $\mathbf{x}, \mathbf{y}$ within the code.

Example. The Hamming distance between the two words

$$12213, 13211$$

is 2, because they disagree in precisely two places. Similarly, the Hamming distance between the two words

$$TOMATO, POTATO$$

is 2, because these two words again disagree in precisely two places.

The "repeat three times" code from earlier has minimum distance 3, because the Hamming distance between 000 and 111 is 3.

Similarly, the second code we described from earlier has minimum distance 3, because every two words in our list disagreed in at least 3 places.

The following theorem explains why we care about this concept of distance:

Theorem. A code C can detect up to s errors in any received codeword as long as $d(C) \geq s+1$. Similarly, a code C can correct up to t errors in any received codeword to the correct codeword as long as $d(C) \geq 2t + 1$.

Proof. If $d(C) \geq s + 1$, then making s changes to any codeword cannot change it into any other codeword, as every pair of codewords differ in at least $s + 1$ places. Therefore, our code will detect an error as long as at most s changes occur in any codeword.

Similarly, if $d(C) \geq 2t+1$, then changing t entries in any codeword still means that it differs from any other codeword in at least $t + 1$ many places; therefore, the codeword we started from is completely unambiguous, and we can correct these errors. □

Example. Using this theorem, we can see that both of our codewords can correct at most one error in any codeword, because their Hamming distances were both three.

Now that we've made this formal, we can now state our question rigorously:

Problem. Suppose that you are given a base q, a length n for your codewords, and a minimum distance d that you want your codewords to be from each other (because you want to be able to correct up to $\lceil (d-1)/2 \rceil$ many errors in any codeword, for example.)

What is the maximum size of C — in other words, what is the maximum information rate you can get a code to have with these parameters?

Amazingly enough, this problem is wide open for tons of values! We really know very little about these maximum values: for example, when $n = 10, q = 2, d = 3$ this question is still open. (I think we know it's between 72 and 79?)

To finish our lecture, we will study a specific case of this problem:

Problem. Take any base q. What is the largest code with all codewords of length 4, such that the minimum distance between any two codewords is 3? (In other words, what's the largest q-ary code set of length 4 that can correct ≤ 1 error in any block of 4?)

Without too much effort, we can get a pretty nice upper bound for our possible codes:

Proposition. If C is a q-ary code of length 4 and minimum distance 3, $|C| \leq q^2$.

Proof. Take any two elements $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ from our code C. Compare (x_1, x_2) and (y_1, y_2). If this pair agreed with each other, these two codewords would be distance 2 from each other, which is impossible because $d(C)$, the minimum distance in our code, is 3. So they must differ.

So: there are q^2 many such pairs to start off any of our codewords, and we've just shown that we cannot repeat any of these pairs. Therefore, there can be at most q^2-many elements in C. $\square$

Surprisingly enough, it turns out that we can use our knowledge of Latin squares to attain this bound!

2 Mutually Orthogonal Latin Squares

Question. Take a deck of playing cards, and remove the 16 aces, kings, queens, and jacks from the deck. Can you arrange these cards into a 4×4 array, so that in each column and row, no two cards share the same suit or same face value?

This question should feel similar to the problem of constructing a Latin square: we have an array, and we want to fill it with symbols that are not repeated in any row or column. However, we have the additional constraint that we're actually putting **two** symbols in every cell: one corresponding to a suit, and another corresponding to a face value.

So: if we just look at the face values, we have a 4×4 Latin square. Similarly, if we ignore the face values and look only at the suits, we should have a different 4×4 Latin square; as well, these two Latin squares have the property that when we superimpose them (i.e. place one on top of the other), each of the resulting possible 16 pairs of symbols occurs exactly once (because we started with 16 distinct cards.)

The generalization of this idea gives us an idea of **orthogonality** for Latin squares, which we define here:

Definition. A pair of $n \times n$ Latin squares are called **orthogonal** if when we superimpose them (i.e. place one on top of the other), each of the possible n^2 ordered pairs of symbols occur exactly once.

A collection of k $n \times n$ Latin squares is called **mutually orthogonal** if every pair of Latin squares in our collection is orthogonal.

Example. The grid of playing cards you constructed earlier if you answered our first question is a pair of 4×4 squares, for the reasons we discussed earlier. To further illustrate the idea, we present a pair of orthogonal 3×3 Latin squares:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} (1,1) & (2,2) & (3,3) \\ (2,3) & (3,1) & (1,2) \\ (3,2) & (1,3) & (2,1) \end{bmatrix}$$

3 Orthogonal Latin Squares and Codes

Proposition 1. *There is a q-ary code of length 4, distance 3, and containing q^2 many elements, whenever there are a pair of mutually orthogonal Latin squares of order q.*

Proof. We leave this for the HW! Instead, we illustrate the proof idea with an example: take two MOLS of order 3.

$$A = \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \hline \end{array}, B = \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 1 & 2 & 0 \\ \hline \end{array}$$

Consider the following construction:

location	s_A	s_B		codewords
$(0,0)$	0	0		0000
$(0,1)$	1	1		0111
$(0,2)$	2	2		0222
$(1,0)$	1	2		1012
$(1,1)$	2	0	$\longrightarrow$	1120
$(1,2)$	0	1		1201
$(2,0)$	2	1		2021
$(2,1)$	0	2		2102
$(2,2)$	1	0		2210

Create a list of all of the 3^2 possible cell locations in a 3×3 grid. For each location, write down the symbol in square A and the symbol in square B: this creates the table at left. If you compress these columns together, you get the list of codewords at the right.

Notice that this table of of codewords has the following property: if you are given any two of the four digits of a codeword, you can uniquely determine which codeword you started with. To see this, simply notice that knowing the row along with either the column, s_A,, or s_B uniquely determines what cell we're talking about (because of the Latin property of both A and B,) and therefore uniquely determines the rest of the values. Similarly, knowing the column and any other position also uniquely determines the cell we're studying, and therefore the codeword. Finally, if you know s_A and s_B, you know all of the other values because this is a pair of mutually orthogonal Latin squares, and therefore (because each pair of symbols shows up exactly once) there is a unique cell corresponding to this pair of symbols.

Therefore, because knowing any two symbols uniquely determines a word, no two words have two locations in common; therefore, the distance between any two words is at least 3. Therefore, $d(C)$ for this code is 3. It is made of base-3 words of length 4, and contains 9 words; therefore, it is exactly what we're looking for in the case that $q = 3$. $\square$