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Lecture 1: Period Three Implies Chaos

Week 1 — *UCSB 2014*

(Source materials: "Period three implies chaos," by Li and Yorke, and "From Intermediate Value Theorem To Chaos," by Huang.)

This lecture, roughly speaking, is about how the intermediate value theorem is a deeply strange and powerful piece of mathematics.

On its first glance, it looks pretty innocuous. Here's the theorem statement, as you've probably seen it in calculus:

Theorem. (Intermediate Value Theorem.) Suppose that f is a continuous function on some interval $[a, b]$, and L is a value between $f(a)$ and $f(b)$. Then there is some value $x \in [a, b]$ such that $f(x) = L$.

On its face, this looks pretty normal, and quite believable: if a continuous function starts at $f(a)$ and ends up at $f(b)$, then it must adopt every value between $f(a)$ and $f(b)$ along the way. Despite its simplicity, the intermediate value theorem has a lot of useful, obvious, and not-entirely-obvious applications:

1. Suppose that you are running a race and are in last place. If you finish in first place, then at some point in time you must have passed the other runners. To make this an intermediate value theorem problem: for each other runner i, let $f_i(t)$ denote the signed distance between you and that runner. At the point in time in which you are in last, the function $f_i(t)$ is negative; at the point in time when you finished the race, $f_i(t)$ is positive. Because f is continuous[1], then there must be some time t where $f_i(t) = 0$, at which point you pass that runner.

2. Suppose that $p(x)$ is a polynomial of odd degree: i.e. that there are coefficients $a_0, \ldots a_n$ such that $p(x) = a_0 + \ldots + a_n x^n$, with n odd and $a_n \neq 0$. Then $p(x)$ has a root: i.e. there is some value x_0 such that $p(x_0) = 0$. This is because for sufficiently large values of x, $p(x)$ will be dominated by its $a_n x^n$ term, and thus become whichever sign a_n is. Therefore, for sufficiently large values of x, $p(x)$ and $p(-x)$ are different signs! So we can apply the intermediate value theorem and choose L to be 0, which gives us that there is some value at which $f(x_0) = 0$.

3. Suppose that $f(x)$ is a continuous function on $[a, b]$, whose range contains the interval $[a, b]$. Then there is some point $x_0 \in [a, b]$ such that $f(x_0) = x_0$: i.e. there is a point in our interval that our function does not change.

   This is not hard to see. Because $[a, b]$ is within the range of $f(x)$, there are two values $c, d \in [0, 1]$ such that $f(c) = 0, f(d) = 1$. If $c = a$ or $d = b$, we've found our point! Otherwise: look at the function $g(x) = f(x) - x$. At c, we have $g(c) = a - c < 0$, because c is a number in $[a, b]$ not equal to a. At d, we have $g(d) = b - d > 0$, because

[1] Assuming that you're not cheating or (less likely) quantum-tunneling during said race.

d is a positive number not equal to b in $[a, b]$. Therefore, by the intermediate value theorem, there is some point x_0 between c and d such that $g(x_0) = 0$. But this means that $f(x_0) - x_0 = 0$; i.e. $f(x_0) = x_0$, and we have our result!

This third example is the one I want to study today, because it allows us to motivate the primary concept we are studying today: the notion of **period.**

1 Periodic Points

Definition. Let $f(x)$ be some function. We say that a point x_0 in the domain of f is a **periodic point** with **period** n if the following two conditions hold:

1. $f^n(x_0)$, the result of applying the function f n times in a row to x_0 (i.e. $\overbrace{f(f(\ldots f}^{\text{n times}}(x_0)\ldots))$, is equal to x_0.
2. For any k, $1 \leq k \leq n-1$, $f^k(x_0) \neq x_0$.

In other words, a point has period n if applying f to that point n times returns that point to itself, and n is the smallest value for which this point returns to itself.

In the third example, we looked at points with period 1. We call such points **fixed points**, because they are **fixed** under the mapping f, and (as we've shown above) it's not too hard to find examples of such objects!

Finding examples of other such points is a bit trickier, but not too hard. Consider the polynomial

$$p(x) = 3x^2 - \frac{7}{2}x + 1.$$

Notice that

- $p(0) = 1$,
- $p(1) = 1/2$, and
- $p(1/2) = 0$;

therefore, 0 is a point of period 3.

Determining whether this function has points with other periods, though: like points with period 5, or 7, or 6 ... seems hard. How can we do this? Well: the intermediate value theorem gave us a way to find fixed points. Perhaps we can build something out of the intermediate value theorem that can find periodic points!

As it turns out, we can do this via the following theorem:

Theorem. Let $f(x)$ be a continuous function on the interval $[a, b]$, and $I_0, \ldots I_{n-1}$ denote a collection of closed intervals that are each contained within $[a, b]$. Assume that

1. $f(I_k) \supseteq I_{k+1}$, for every $k = 0 \ldots n-2$, and

2. $f(I_{n-1}) \supseteq I_0$,

where by $f(I_k)$ we mean the set given by applying f to all of the points in the interval I_k. (In other words, applying f to any one interval I_k gives you a set that contains the next interval I_{k+1})

Then there is some point $x_0 \in I_0$ such that

1. $f^n(x_0) = x_0$, and
2. $f^k(x_0) \in I_k$, for every $k = 0, \dots n-1$.

Note that if we can make all of the I_k's for $k \geq 1$ not contain points in I_0, then any solution of the above is a point with period n, because each $f^k(x_0)$ will be contained in I_k, and therefore not a point in I_0 (and in particular not equal to x_0 itself!)

We prove this theorem here:

Proof. We start by observing the following useful fact:

Lemma 1. *If $f(I_k) \supseteq I_{k+1}$, then there is a subinterval of I_k such that $f(I_k) = I_{k+1}$.*

Proof. This is a consequence of the intermediate value theorem. Suppose that $I_{k+1} = [c, d]$, for some pair of endpoints c, d. Because $f(I_k) \supseteq [c, d]$, there are values that get mapped to c and d themselves. Pick x_1, x_2 such that $f(x_1) = c, f(x_2) = d$, and x_1, x_2 are the closest two such points with this property.

Claim: this means that $f([x_1, x_2]) = [c, d]$. To see why, simply use the intermediate value theorem to see that $f([x_1, x_2])$ contains $[c, d]$. Moreover, if it contained a point $z \notin [c, d]$, then (if x_3 maps to z) the intermediate value theorem would tell us that we can find a point that maps to one of c, d in one of the intervals $[x_1, x_3], [x_3, x_2]$, in such a way that violates our "closest two points" property! So we've proven our lemma.

$\square$

Given this lemma, our proof is relatively simple. Find intervals I_k^* such that

- $I_0^* \subseteq I_0$ and $f(I_0^*) = I_1$,
- $\dots$
- $I_{n-2}^* \subseteq I_{n-2}$ and $f(I_{n-2}^*) = I_{n-1}$, and
- $I_{n-1}^* \subseteq I_{n-1}$ and $f(I_{n-1}^*) = I_0$.

Then, as a consequence, we must have that

$$f^k(I_0^*) = I_k^*,$$

for any k, and

$$f^n(I_0^*) \subseteq I_0^*.$$

To finish our proof, then, we just have to notice that because f is continuous, so is f^n! Therefore, because $f^n(I_0^*) \subseteq I_0^*$, we know from our result at the start of the lecture on fixed points that there is some $x_0 \in I_0^*$ such that $f^n(x_0) = x_0$! Therefore we've proven our claim: we have found a point x_0 such that

1. $f^n(x_0) = x_0$, and
2. $f^k(x_0) \in I_k$, for every $k = 0, \ldots n-1$.

$\square$

2 Why We Care: Chaos

So: the reason we care about all of this isn't really because we want to find periodic points; rather, it's because we want to know when we can avoid them! Consider the following problem:

Problem. Suppose we have a fluid filled with particles in some reasonably-close-to-one-dimensional object, which we can model as an interval $[a, b]$. Furthermore, suppose that we know how this fluid is "mixing:" i.e. that we have some function $f : [a, b] \to [a, b]$, such that $f(x)$ tells you where a particle at location x will wind up after one step forward in time.

Where do your particles go? Do they settle down? Do they all clump together at one end? In other words: what does f^n look like as n grows very large?

Something you might hope for is that your fluid particles settle down: that they either converge to various states, or at least that they all settle into some small set of predictable periodic orbits. In the worst case scenario, however, you might have something like the following:

Definition. A function f is called **chaotic** if for any n, it has a particle of period n.

So! The punchline for this class is the following theorem of Li and Yorke:

Theorem. Suppose that f is a continuous function on $[a, b]$ with range contained in $[a, b]$. Then if f has a 3-periodic point, it is chaotic.

Proof. Take a triple $x_0 < x_1 < x_2$ of points that form a 3-periodic orbit. Either $f(x_1) = x_2$ or $f(x_1) = x_0$; assume that $f(x_1) = x_0$ without loss of generality, as the proof will proceed identically in the other case. Then we have $f(f(x_1)) = f(x_0) = x_2$.

Let $I_0^\star = [x_0, x_1]$ and $I_1^\star = [x_1, x_2]$. Note that because $f(x_0) = x_2, f(x_1) = x_0, f(x_2) = x_1$, by the intermediate value theorem, we have

- $f(I_0^\star) \supseteq [x_0, x_2] \supset I_1^\star, I_0^\star$, and
- $f(I_1^\star) \supseteq [x_0, x_1] \supset I_0^\star$.

So: let $I_0 = \ldots I_{n-2} = I_0^\star$, and $I_{n-1} = I_1^\star$. Apply our theorem from before that was designed to find periodic points: this gives us a point x_0 such that $x_0, f(x_0), \ldots f^{n-2}(x_0) \in I_0^\star$, $f^{n-1}(x_0) \in I_1^\star$, and $f^n(x_0) = x_0$.

I claim that this point is a n-periodic point. We already have that $f^n(x_0) = x_0$; we just need to prove that $f^k(x_0) \neq x_0$, for any $k = 1, \ldots n-1$. To see this, proceed by contradiction. Suppose that $f^k(x_0) = x_0$, for some k. Then $f^{n-1}(x_0)$ is equal to an earlier term $f^{n-1-k}(x_0)$, because applying f k times is the same thing as doing nothing. But this means that

- on one hand, $f^{n-1}(x_0) \in I_1^\star$, and
- on the other hand, $f^{n-1}(x_0) = f^{n-1-k}(x_0) \in I_{n-1-k} = I_0^\star$.

Therefore this point is in both sets. But the only point in both $I_0^\star = [x_0, x_1]$ and $I_1^\star = [x_1, x_2]$ is x_1; so $f^{n-1}(x_0) = x_1$. But then $f^n(x_0) = x_2$, which is not in $I_0^\star$ and therefore in particular is not x_0!

So we have a contradiction to our assumption that x_0 was not a point with period n.

□

This is ... weird. All we used in the above statement was that there was a point with period 3 – i.e. some point such that $f(f(f(x))) = x$, while $f(x), f(f(x) \neq x$. And out of nowhere we got points of **every** period: chaos!

Cool, right?