Math 7h	Professor: Padraic Bartlett

Homework 5: Sizes of Infinity

Due Tuesday, week 6, at the start of class — *UCSB 2014*

Try one or more of the following problems! (If you were in this lecture in the fall-quarter run of this course, try a problem that is new or that you did not try before!)

1. Can there ever be more words than numbers?

   Specifically: let's suppose that we're limiting ourselves to the 26-character Latin alphabet, and that the only kinds of things that can be **words** are finite strings of characters from the Latin alphabet. So things like

   - rabbit
   - barglearglesnarg
   - ssss
   - froyo

   are all possibly words. Call the set of all possible words $\mathbb{W}$. Is the set $\mathbb{W}$ the same cardinality as $\mathbb{N}$?

2. Define the **Cantor set** $\mathcal{C}$ as follows:

   - Start with the interval $[0, 1]$. Call this set C_0.
   - Remove the middle-third of this set, so that you have $[0, 1/3]$ and $[2/3, 1]$ left over. Call this set C_1.
   - Remove the middle-third of those two sets, so that you have $[0, 1/9], [2/9, 1/3], [2/3, 7/9], [8/9, 1]$ left over. Call this set C_2.
   - Repeat this process!

   Define $\mathcal{C}$, the Cantor set, as the set made by taking all of the elements x such that x is in C_i, for every i.

   (a) Find an element in $\mathcal{C}$.

   (b) Show that $\mathcal{C}$ contains infinitely many elements.

   (c) Can you make a bijection between $\mathcal{C}$ and $[0, 1]$?

3. Given a set A, we say that a set B is a **subset** of A if every member of B is also a member of A. For example, if $A = \{w, x, y, z\}$ and $B = \{w, z\}$, then B is a subset of A, because every member of B is also a member of A. Conversely, the set $C = \{x, y, z, \alpha\}$ is not a subset of A, because the element α is a member of C and not a member of A.

   A special set that bears mentioning is the **empty set**, denoted $\emptyset$. This is the set that contains no elements.

   Given a set A, the **power set** of A, denoted $\mathcal{P}(A)$, is the collection of all of the subsets of A. For example, the power set of $\{1, 2, 3\}$ is the set containing the following eight sets:

- $\emptyset$
- $\{1\}$
- $\{2\}$
- $\{3\}$
- $\{1,2\}$
- $\{1,3\}$
- $\{2,3\}$
- $\{1,2,3\}$

We write this as

$$\mathcal{P}(A) = \Bigg\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\Bigg\}.$$

(It may seem weird for a set to contain other sets, but this is entirely valid! Sets are just collections of objects, and those objects can be all sorts of strange things, including other sets. If this bothers you, come and talk to me at office hours, or send me an email!)

(a) Show that the empty set is a subset of every set.

(b) Calculate the power set of the following three sets:

i. {thyme, sage}

ii. {cat, dog, pheasant, quail}

iii. $\{\alpha, \beta, \gamma, \delta, \epsilon\}$.

How many elements are in each power set?

(c) Prove that for any set A, the cardinality of the power set of A, $|\mathcal{P}(A)|$ is greater than the cardinality of A, $|A|$.

4. Create an injection from $\mathcal{P}(\mathbb{N})$ to the real numbers $\mathbb{R}$.