Math 7h	Professor: Padraic Bartlett

Lecture 3: Graph Theory

Week 3	*UCSB 2013*

Mathematicians like to use graphs to describe lots of different things. Groups, electrical networks, airplane routes, and the internet itself are all objects which graphs are used to model; consequently, mathematicians have came up with several different definitions for the word "graph" itself! We state several of these definitions here:

1 The Basics

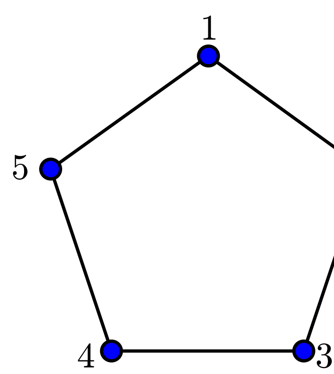
Definition. A **simple** graph G with n vertices and m edges consists of the following two objects:

1. a set $V = \{v_1, \ldots v_n\}$, the members of which we call G's **vertices**, and
2. a set $E = \{e_1, \ldots e_m\}$, the members of which we call G's **edges**, where each edge e_i is an unordered pair of distinct elements in V, and no unordered pair is repeated. For a given edge $e = \{v, w\}$, we will often refer to the two vertices v, w contained by e as its endpoints.

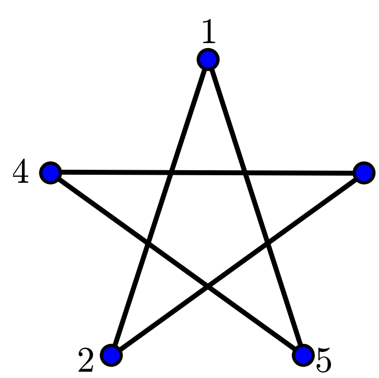
Example. The following pair (V, E) defines a simple graph G on five vertices and five edges:

- $V = \{1, 2, 3, 4, 5\}$,
- $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\}$.

Something mathematicians like to do to quickly represent graphs is **draw** them, which we can do by taking each vertex and assigining it a point in the plane, and taking each edge and drawing a curve between the two vertices represented by that edge. For example, one way to draw our graph G is the following:



However, this is not the only way to draw our graph! Another equally valid drawing is presented here:



As mentioned before, there are other possible definitions of a graph:

Definition. A **simple directed** graph G with n vertices and m edges consists of the following two objects:

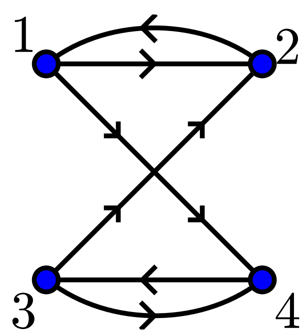
1. a set $V = \{v_1, \ldots v_n\}$ of vertices, and
2. a set $E = \{e_1, \ldots e_m\}$ of edges, where each edge e_i is an **ordered** pair of distinct elements in V, where no ordered pair is repeated.

The only difference between this definition and the definition for simple unordered graphs is that all of our edges have an ordering – i.e. the edge (a, b) is different from the edge (b, a).

Example. The following pair (V, E) defines a simple directed graph G on four vertices and six edges:

- $V = \{1, 2, 3, 4\}$,
- $E = \{(1, 2), (2, 1), (3, 4), (4, 3), (1, 3), (4, 2)\}$.

We can draw such a graph in the exact same method as before, provided that we put little arrows on our edges to indicate which direction they're traveling:



As mentioned before, there are other possible definitions of a graph:

Definition. A **multigraph** graph G with n vertices and m edges consists of the following two objects:

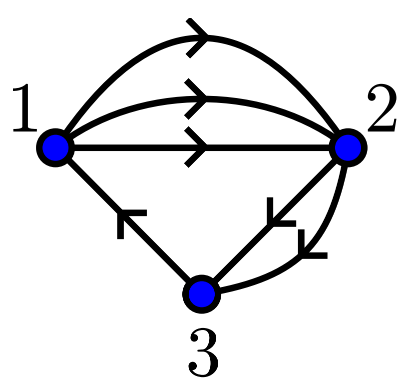
1. a set $V = \{v_1, \ldots v_n\}$ of vertices, and
2. a set $E = \{e_1, \ldots e_m\}$ of edges, where each edge e_i is an **ordered** pair of elements in V.

Note that in this definition we allow edges to be repeated, and furthermore that we allow an edge to contain the same element twice.

Example. The following pair (V, E) defines a multigraph graph G on three vertices and six edges:

- $V = \{1, 2, 3\}$,
- $E = \{(1, 2)_1, (1, 2)_2, (1, 2)_3, (2, 3)_4, (2, 3)_5, (3, 1)_6\}$.

We can realize this graph as the following picture:



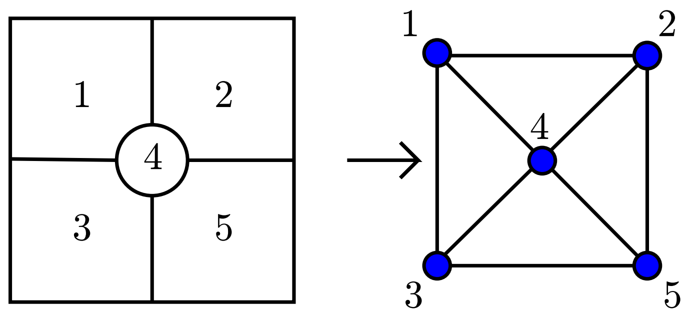
Typically, when a mathematician simply refers to a graph, they will mean a **simple** graph. Throughout this course, we will usually work with simple graphs; if we want to refer to any of the other concepts for graphs, we will explicitly say that we're doing so.

2 Graphs as Models

Before we start delving into the theory, we first show some of the ways in which graphs can model some remarkably interesting problems:

Example. (Maps.) Suppose we start with a map M consisting of several countries, and we want to find a way of assigning each country a color so that no two countries sharing a border are the same color. How many colors do we need to do this? Well, consider the following simple graph we can make out of our map, where we set

- V = the collection of countries on our map, and
- $E = \{\{a, b\} : a$ and b are countries that share a border.$\}$



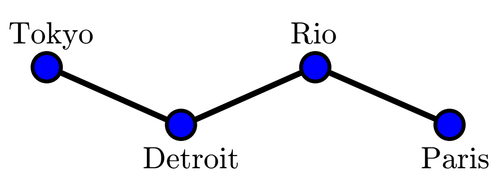
Then, our question is the following: given any graph that we can get from a map, how many colors do we need to color its vertices so that no edge connects two vertices of the same color?

The answer to this question is 4, and the problem itself is known as the **Four-Color Theorem** – proven in 1976, it is one of the first problems in mathematics to have been resolved with a computer. To this day, there are no non-computer-aided proofs known of the four-color theorem.

Example. (Travel.) Suppose you're a travelling salesman, going between cities in the country to sell your product. Specifically, suppose you have a list of C cities that you have to make it to, F a list of flights between these cities, and you're starting in some city $c \in C$. Is there some flight path you could take that would make it so you never had to visit any city twice?

Well, this clearly depends on your lists C and F, and where you're starting from. For example, if your cities were {Tokyo, Detroit, Rio, Paris} and your flights were {{ Tokyo, Detroit}, { Detroit, Rio}, {Rio, Paris }}, you could do this if you were starting in Tokyo or Paris, but not if you started in Detroit or Rio.

A natural way to visualize this problem is with graph theory! Specifically, if we think of C as the set of vertices and F as the set of edges, we can visualize such a travel map as follows:



Phrased in this fashion, our question is the following: given a graph G, is there a **path**[1] that visits every vertex exactly once?

Such a path is called a Hamiltonian path, and the question of whether such paths exist on an arbitrary graph is NP-complete[2].

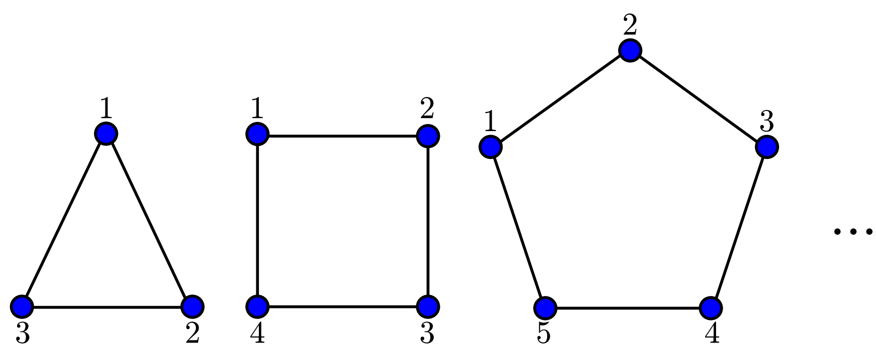
[1] A **path** of length n is a sequence of alternating vertices and edges $v_0, e_{01}, v_1, e_{12}, \ldots v_n$ from our graph G, so that each edge $e_{k,k+1}$ connects the vertex v_k to the vertex v_{k+1}. Intuitively, a path is just a way of "walking around" on our graph for n steps.

[2] This means, roughly, that there is no "fast" way to find out whether such a path exists; basically, there isn't any way to find out if a graph has a Hamiltonian path that's much faster than just trying out *every* path and seeing if any of them work. Furthermore, if you *could* find a faster algorithm in general, the fact that this problem is NP-complete would allow you to solve a ton of other really difficult problems (like how to factor numbers into primes) quickly as well. See Wikipedia for a better description of what NP-complete means.

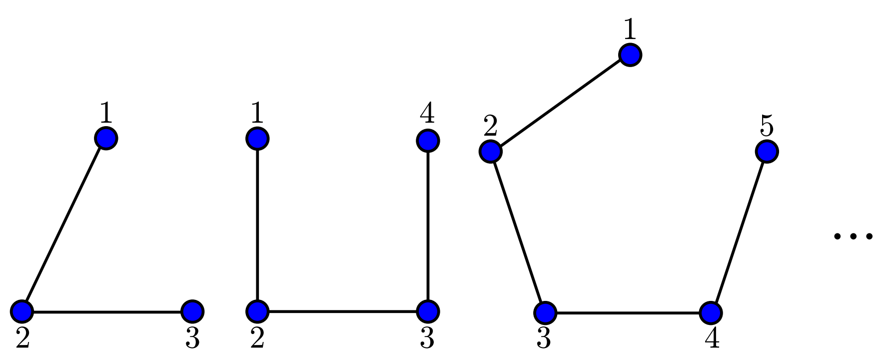
3 Several Key Graphs

In the field of graph theory, there are several graphs which come up enough to where we've given them names. We list them here, along with some of their properties:

- **The cycle graph** C_n**.** The cycle graph on n vertices, C_n, is the simple graph on the vertex set $\{v_1, v_2, \ldots v_n\}$ with edge set $E(C_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$. The cycle graphs C_n can be drawn as n-gons, as depicted below:

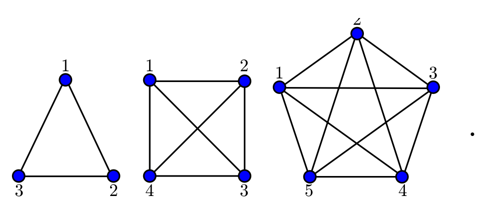


- **The path graph** P_n**.** The path graph on n vertices, P_n, is the simple graph on the vertex set $\{v_1, v_2, \ldots v_n\}$ with edge set $E(C_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots \{v_{n-1}, v_n\}\}$. The path graphs P_n can be drawn as paths of length n, as depicted below:



  Every vertex in a P_n has degree 2, except for the two endpoints v_1, v_n, which have degree 1. P_n contains $n - 1$ edges.

- **The complete graph** K_n**.** The complete graph on n vertices, K_n, is the simple graph on the vertex set $\{v_1, v_2, \ldots v_n\}$ that has every possible edge: in other words, $E(K_n) = \{\{v_i, v_j\} : i \neq j\}$. We draw several of these graphs below:



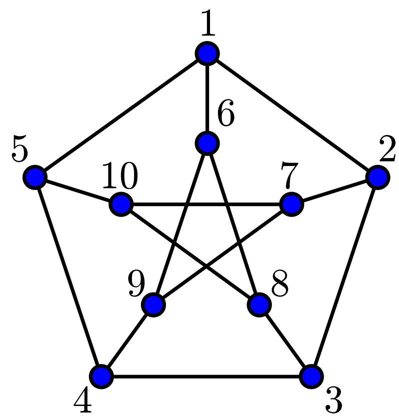
  Every vertex in a K_n has degree $n - 1$, as it has an edge connecting it to each of the other $n-1$ vertices; as well, a K_n has $n(n-1)/2$ edges in total in it, by the degree-sum formula. (Explicitly: every vertex has degree $n - 1$ and there are n vertices, therefore the sum of the degrees of K_n's vertices is $n(n - 1)$. We've shown that this quantity is twice the number of edges in the graph; dividing by 2 then tells us that the number of edges in K_n is $n(n - 1)/2$, as claimed.)

- **The complete bipartite graph** $K_{n,m}$**.** The complete bipartite graph on $n + m$ vertices with part sizes n and m, $K_{n,m}$, is the following graph:

  - $V(K_{n,m}) = \{v_1, v_2, \ldots v_n, w_1, w_2, \ldots w_m\}$.
  - $E(K_{n,m})$ consists of all of the edges between the n-part and the m-part; in other words, $E(K_{n,m}) = \{(v_i, w_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$.

  The vertices v_i all have degree m, as they have precisely m edges leaving them (one to every vertex w_j); similarly, the vertices w_j all have degree n. By either the degree-sum formula or just counting, we can see that there are nm edges in $K_{n,m}$.

- **The Petersen graph** P The Petersen graph P is a graph on ten vertices, drawn below:



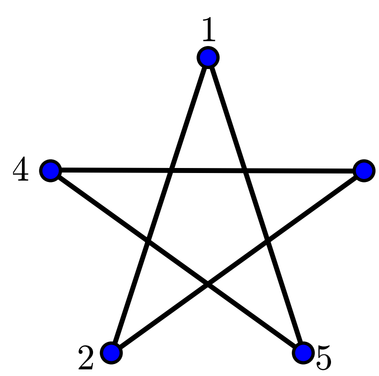
The vertices in P all have degree three; by counting or the degree-sum formula, P has 15 edges.

4 The Degree-Sum Formula

Graphs in general – as you may have noticed – can be remarkably hairy and complicated things. As such, we might wonder whether we can say anything about a general graph at all; in other words, if G is a graph, can we say anything that restricts G?

As it turns out, we can! Consider the following definition:

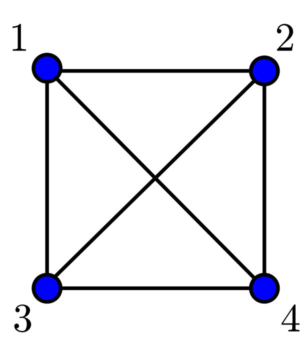
Definition. In a simple graph G, we say that a vertex $v \in V(G)$ has **degree** k iff there are exactly k edges in $E(G)$ that involve v. For example, in the graph drawn below, every vertex has degree 2:



One quick question we could ask: for what values of n can we make a graph with the degrees of all vertices $= 3$?

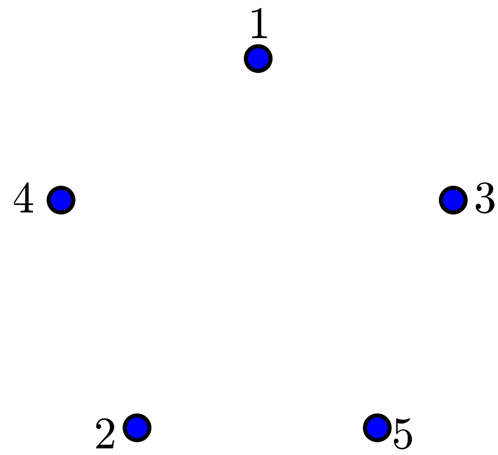
Well: in order to have each vertex have three neighbors[3], we need to have at least four vertices (as each vertex needs three other vertices to send edges to.)

In the case where $n = 4$, we can make every vertex have degree 3, via the graph below:



How about for $k = 5$? Try it for yourself, on the vertices below:

[3]We say that the vertices v and w are neighbors iff there is an edge connecting v and w. We denote the set of all neighbors of v as $N(v)$.



As you may have noticed, no matter what you try, you can't get all of the degrees to be three! As it turns out, this is completely impossible – there is no graph on 5 vertices for which all of the vertices have degree 3!

In general, much more is true:

Theorem 1. *(Degree-Sum Theorem:) For a graph G on n vertices, with m edges and vertex set $V = \{v_1, \dots v_n\}$,*

$$\sum_{i=1}^{n} \deg(v_i) = 2 \cdot m.$$

Proof. The sum on the left-hand side is adding up all of the degrees of vertices in our graph. But the degree of each vertex v_i is just the number of edges that involve v_i; because every edge uses precisely two vertices, every edge is counted exactly twice on the left-hand side. Therefore, we have that this sum must be twice the number of edges. □

Corollary 2. *Every graph must have an even number of vertices of odd degree; in other words, a graph cannot have an odd number of vertices of odd degree.*

As a very specialized case of the above corollary, we've proven that that there is no graph on five vertices where all vertices have degree three.