Math 7h | Professor: Padraic Bartlett

Homework 3: Graph Theory

Due 10/22/13, at the start of class | *UCSB 2013*

Instructions: Choose **one** of the problems below, and work on it until either:

1. You solve the problem, or
2. You have spent about 90 minutes working seriously on the problem.

Homework Problems

1. ($*$) A graph G is called **Eulerian** if it contains a path P that satisfies the following two properties:

   - P starts and ends on the same vertex.
   - P uses every edge in G exactly once.

   Show that a graph G is Eulerian if and only if the degree of every vertex in G is even.

2. A **graceful labeling** of a graph with E edges is a labeling $l(v)$ of its vertices with distinct integers from the set $\{0 \ldots E\}$, such that each edge $\{u, v\}$ is uniquely determined by the difference $|l(u) - l(v)|$. A graph is called **graceful** if it has a graceful labeling.

   Show that all paths P_n are graceful.

3. Show that K_n is graceful if and only if $n \leq 4$.

4. A **tree** is a graph that does not contain any copies of the cycle graphs C_n, and is also connected (i.e. given any two vertices in our graph, there is a path of edges that goes from the first vertex to the second.)

   Show that all trees are graceful.

5. Consider the game on six vertices we played at the end of class:

   - Set-up: a chalk board with six vertices drawn on it; two players, one with a piece of red chalk, the other with a piece of blue chalk.
   - Players alternate drawing edges between vertices on our board. When drawing edges, no player can draw an edge that has been drawn on a previous turn.
   - A player loses if they ever complete a triangle made entirely out of the color of chalk they are using.

   Show that no matter how these two players play, someone will eventually lose: i.e. that there is no way to color the edges of K_6 red and blue in such a way that there is no monochromatic triangle.