Math 137B	Professor: Padraic Bartlett

Lecture 5: Planar and Nonplanar Graphs

Week 7	*UCSB 2014*

(Relevant source material: Chapter 6 of Douglas West's Introduction to Graph Theory; Section V.3 of Béla Bollobás's Modern Graph Theory; various other sources.)

In last week's class, we proved that the graphs K_5 and $K_{3,3}$ were **nonplanar**: i.e. that there was no way to draw these graphs in the plane without having some of their edges cross. In this week's lectures, we are proving that those two graphs, in a sense, are the only obstructions that can stop a graph from being planar!

We formalize this here with the following two definitions:

Definition. We say that a graph G is a subdivision of a graph H if we can create H by starting with G, and repeatedly replacing edges in G with paths of length n. We illustrate this process here:

Definition. A graph is called **Kuratowski** if it is a subdivision of either K_5 or $K_{3,3}$.

We know that a graph cannot be planar if it contains a Kuratowski subgraph, as those subgraphs are nonplanar. As stated above, our goal is to prove that these necessary conditions are also sufficient:

Theorem. (Kuratowski's theorem) A graph is planar if and only if it does not contain a Kuratowski graph as a subgraph.

To perform a proof of this kind — characterizing all graphs with a given property as having some special kind of substructure — a natural first step is to simply start exploring what nonplanarity **looks like** in general. What kinds of graphs are nonplanar? What kinds of properties are easy to observe about nonplanar graphs?

We start this exploration here. To simplify matters somewhat, we are going to work with the following definition when considering nonplanar graphs:

Definition. We say that a graph G is **minimally nonplanar** if G is nonplanar, but any proper subgraph of G is planar.

The idea here is that minimality makes our lives a lot easier (as it did when we were working with graph colorings and several other properties), as it allows us to deduce certain properties about all of the subgraphs of our given graph! At the same time, we don't "lose" any generality by working with minimally nonplanar graphs; because the result we are trying to prove is one about certain subgraphs that all nonplanar graphs have, it suffices to simply show that all minimally nonplanar graphs have those subgraphs, as every nonplanar graph has a minimally nonplanar subgraph (find the smallest nonplanar subgraph!)

We make our first observation about such graphs here:

Proposition. Any minimally nonplanar graph G is at least 2-connected[1].

Proof. This is a fairly short and elegant proof by contradiction. Assume not; i.e. that G is ≤ 1-connected, and in particular that there is some vertex x that we can delete from G that breaks G into multiple components, each one of which is disconnected from the others. Call these components $G'_1, \ldots G'_n$, and let $G_1 = G'_1 \cup \{x\}, \ldots G_n = G'_n \cup \{x\}$.

Because G is minimally nonplanar, all of these graphs $G_1, \ldots G_n$ are planar, as they are each individually proper subgraphs of G.

Now, recall from last week our observation that if G is a planar graph and F is any face of G, we can draw G in the plane in such a way that F is the outside face of G. Apply this observation to each of the $G_1, \ldots G_n$ and the face in each graph that contains x.

Finally, take any one of these G_i's along with this planar embedding with x on the outside face. By translation, place x at the origin in the plane, and by stretching and compressing space, squish all of G_i so that it fits within a wedge with angle at most $2\pi/n$ centered at the origin.

By doing this with all of the G_i's and placing each in its own wedge centered at the origin, we have created a planar embedding of G! This contradicts our claim that G was nonplanar; therefore, our assumption that G was not at least 2-connected must have been false. □

Our eventual goal is to prove that any nonplanar graph must contain a Kuratowski subgraph. However, a direct proof of this seems tricky; what if we attempt contradiction instead? In other words, suppose that we take a minimally nonplanar graph that contains no Kuratowski graph; can we create a contradiction? At the least, can we find some sorts of properties that such a graph would have to have, that we could perhaps use to find a contradiction later?

We find one of these properties in the following proposition:

Proposition. Suppose that G is a nonplanar graph, G contains no Kuratowski subgraphs, and that G is the smallest graph in terms of numbers of edges amongst all such graphs. Then G is at least 3-connected.

Proof. We again proceed by contradiction. Take any such graph G. We know from our earlier proposition that this graph is at least 2-connected; let's assume for contradiction that it is exactly two-connected, and therefore that there is some pair of vertices x, y such that deleting x and y from our graph disconnects it.

Name each of these disconnected components $G'_1, \ldots G'_n$. Define $G_1 = G'_1 \cup \{x, y, e_{xy}\}, \ldots G_n = G'_n \cup \{x, y, e_{xy}\}$; i.e. take each of the disconnected components $G'_1, \ldots G'_n$, and add to each the two vertices x, y and the edge e_{xy} connecting x to y. Notice that we do this even if the edge e_{xy} was not in our original graph!

[1] A graph G is k**-connected** if you can remove k vertices from G in such a way that makes G disconnected, but there is no way to remove less than k vertices from G in such a way that G is disconnected. Beware: many authors will use k-connected to denote "at least k-connected," i.e. the weaker claim that there is no way to disconnect G by removing less than k vertices.

We do this because it gives us the following observation: at least one of these graphs G_i must be nonplanar! To see this observation, proceed by contradiction, and assume not. Then each G_i is planar, and can be drawn in the plane in such a way that the face containing the edge e_{xy} is on the outer face of this graph. (This is where we needed this edge to exist; without it, we might not be able to draw G_i in the plane with both x and y on the outer face.)

Take G_1, and draw it in the plane in such a fashion. Now, take G_2, and expand G_2 / squish G_1 so that the inner face containing e_{xy} contains G_1 and overlaps with x, y. Repeat this process with the remaining $G_3, \ldots G_n$ graphs, creating a sort of Matryoshka-doll sequence of nested planar graphs:

In particular, notice that the result of this process is a planar graph, which contradicts our assumption that G was nonplanar! This verifies our observation that there is some G_i that is nonplanar, as desired.

Take this G_i. Because it is smaller than our original graph and is nonplanar, it must contain a Kuratowski graph. If this Kuratowski subgraph does not use the edge e_{xy}, or if the edge e_{xy} is in G, then this Kuratowski subgraph is a subgraph of G, which contradicts our claim. Otherwise, e_{xy} is used in this Kuratowski subgraph, and furthermore is not in G!

In this case: take any other component G_j, and some vertex $z \in G_j'$. Because G is 2-connected, there must be a path from z to x and to y in G_j, as otherwise we could simply delete one of x, y to disconnect z from G_i. But this means that there is a path from x to y that goes through G_j! If we replace e_{xy} with this path in our Kuratowski subgraph, we have simply replaced an edge with a subdivision of that edge, which does not change the Kuratowski property: so our graph G has a Kuratowski subgraph, which is a contradiction.

$\square$

We now know that minimally nonplanar + no Kuratowski graphs implies at least 3-connected; what happens if we drop the nonplanarity result? How does at least 3-connected interact with the idea of simply not having Kuratowski subgraphs? Again, if this feels somewhat unmotivated, what we're doing throughout this paper is a sort of "proof by exploration:" we're taking two properties we want to understand (nonplanarity, no Kuratowski subgraphs) and basically just bashing them into all of the other properties we have until we get someting promising.

In many cases, our choice of those other properties is mostly at random: there isn't a particularly great reason for looking at connectivity other than that it is often a useful thing to consider for all graphs. I.e. if you were attempting to solve this problem on your own on a desert island, you would probably have also looked at lots of things like chromatic/clique numbers, max/min degree, etc. on the way; the only reason those explorations aren't in this writeup is that they don't lead to the proof we're searching for! In general, this is the difference between your typical lecture/textbook and one's own research; the first presents a clean and orderly path from A to Z, while the second consists of paths from A to

$B, C, D, E, F, \ldots$, with maybe Z showing up at the end if the author is lucky. So, if you're reading these notes and wondering "why," the answer is simply "because all of the other things didn't work."

However, in other cases, there are very natural definitions and concepts to consider and work with. In particular, if we are working with subdivisions of graphs, the concept of "contraction" — in a sense, the opposite of subdivision — is a handy one to have:

Definition. Take a graph G, and an edge $\{x, y\} = e \in G$. The graph $G \cdot e$, formed by "contracting" e, is the graph formed by taking G, deleting the edge e, and identifying the two endpoints x, y of e.

We draw an example of this this below:

We first show that if we are careful about our contractions, then we can preserve the property of "at least 3-connectedness."

Proposition. If G is a graph on at least 5 vertices that is at least 3-connected, then there is some edge $e \in G$ such that $G \cdot e$ is at least 3-connected.

Proof. Again, we proceed by contradiction. Assume not: that for every edge $e = \{x, y\}$ that we contract, the resulting graph $G \cdot e$ is 2-connected. Notice that if this happens, the 2-disconnecting set in $G \cdot e$ must use the newly-created vertex v_e, as otherwise this set would also disconnect G itself! If this 2-disconnecting set is $\{v + e, z\}$, call z the "mate" of$\{x, y\}$; in this sense, we are assuming for a contradiction that every edge $\{x, y\}$ has a mate z.

Notice that in our original graph G, deleting the two vertices corresponding to any edge $e = \{x, y\}$ along with their mate z must disconnect our graph, because deleting v_e, z from $G \cdot e$ disconnects this graph.

When we delete such a triple x, y, z from G, we get a collection of disconnected components $G_1, \ldots G_n$ of various sizes. Assign to each edge the size of the largest connected component that arises from deleting that edge's vertices along with its mate.

Let $\{x, y\}$ denote the edge with the largest such number assigned to it, let z be the mate of this edge, and let G_1 be this "largest component."

Let G_2 be some other non-G_1 component that we get after deleting x, y, z from G. We know that there must be an edge from z to some other vertex $v \in G_2$, as otherwise deleting simply x and y from G would separate G_2 from the rest of the graph, and G itself is not 2-connected. Let w be the mate of the edge $\{z, v\}$. What can we say about w?

On one hand, we can observe that w cannot be a part of any G_i set for $i \neq 1$; if it were, then deleting z, v, w would yield a decomposition of G into pieces, one of which would contain all of G_1 along with the two vertices x, y! Because we picked $\{x, y\}$ to get the largest possible component G_1, this is impossible.

On the other hand, if w is in G_1 then deleting z, v, w does not actually disconnect our graph! This is because of the following observations:

1. Take any vertex a in our graph, in any part G_i. There must be a pair of vertex-disjoint paths from a to both x and y, because if there wasn't we could disconnect a from other vertices in our graph by deleting w and at most one vertex on such a path.

2. In particular, this tells us that we take any part G_i and delete at most one vertex from it, there is a path from any vertex a in that part to one of x, y.

3. But this means that the graph $G \setminus \{v, w, z\}$ is connected: given any vertex a in any part G_i, we can find a path from a to one of x, y, and from there go to any other vertex b in any other part G_j.

This gives us the desired contradiction, and shows that there must be some edge e in our graph such that $G \cdot e$ is still 3-connected. □

We now show that no matter how we contract edges, we must preserve the property of not containing a Kuratowski subgraph:

Proposition. If G does not have a Kuratowski subgraph, then neither does $G \cdot e$.

Proof. It is easier to work with the contrapositive of this statement, as it gives us a positive claim to work with (i.e. $G \cdot e$ contains this set structure, which is a concrete starting point to work from) rather than the arguably more-difficult to understand negative claim (G does not contain some set structure, which does not obviously tell us a lot about what G **does** contain.) Specifically, we will show that if $G \cdot e$ contains a Kuratowski subgraph, then so does G itself.

Take any graph G and edge e such that $G \cdot e$ contains a Kuratowski subgraph. If the vertex v_e that we got by contracting this edge is not a part of this Kuratowski subgraph, then G trivially contains whatever Kuratowski subgraph is contained within $G \cdot e$.

The interesting cases, then, occur when v_e is a part of this Kuratowski subgraph. First, notice that in any Kuratowski graph, the degree of any vertex is at least 2 and at most 4; this is because we have started with either a K_5 or $K_{3,3}$ and repeatedly subdivided, which is a process that does not change the degree of any already-existing vertices in the target graph and introduces new vertices of degree 2.

In the event that v_e is a vertex of degree at most 2 inside of this Kuratowski subgraph, there are (up to labeling) only two possibilities for what $\{x, y\}$ could have been in the original graph:

1. Both of the Kuratowski subgraph edges could have had the same vertex x as their endpoint.

2. One Kuratowski subgraph edge could have had x as its endpoint, and the other could have had y as its endpoint.

In either case, this Kuratowski subgraph is still a Kuratowski graph when expanded back out to G, with potentially a new subdivided edge.

If v_e is a vertex of degree 3 inside of the Kuratowski subgraph, we can use similar logic to see that one of the following two cases up to labeling must hold:

1. All three of the Kuratowski subgraph edges could have had the same vertex x as their endpoint.
2. Two Kuratowski subgraph edge could have had x as its endpoint, and the other edge could have had y as its endpoint.

Again, in either case, this Kuratowski subgraph is still a Kuratowski graph when expanded back out to G.

The case when v_e is a vertex of degree 4 has a similar set of potential cases:

1. All four of the Kuratowski subgraph edges could have had the same vertex x as their endpoint.
2. Three Kuratowski subgraph edge could have had x as its endpoint, and the other edge could have had y as its endpoint.
3. Two Kuratowski subgraph edges could have x as an endpoint, while the other two could have y as their endpoint.

The first two cases are again still obviously subdivisions of the same Kuratowski graph. The third, however, is different! We have split our vertex of degree 4 into two vertices of degree 3, and we cannot simply get rid of any of these degree-3 vertices by arguments about subdivision (as subdivision only creates vertices of degree 2!)

Instead, something much more beautiful happens. First, notice that if v_e was a vertex of degree 4, then we were in a subdivided K_5, as subdivided $K_{3,3}$'s do not have vertices of degree 4 in them! Therefore, we are actually in the following situation

The graph on the right: a subdivided $K_{3,3}$! On one hand, this proves our claim: no matter what G was, if $G \cdot e$ has a Kuratowski subgraph, we have shown that G must as well. On the other hand, we've actually seen a nice connection between K_5 and $K_{3,3}$ that illustrates how these two graphs are related. □

By combining our results thus far, we have the following: given any at least 3-connected graph without Kuratowski subgraphs, we can repeatedly contract edges in this graph in a way that preserves these two properties! This is somewhat surprising, as when we take any at least 3-connected / no-Kuratowski subgraph and reduce it all the way down to 4 vertices, we know that it is K_4 (as that's the only 3-connected graph on four vertices,) and in particular that this graph is **planar**!

This leads us to hope that an inductive proof might tell us that **any** at least 3-connected graph without Kuratowski graphs must be planar; i.e. if when we reduce such graphs down, we eventually get a planar graph, then perhaps when we work our way back up we must still have a planar graph! This turns out to work, as described here:

Proposition. If G is at least 3-connected and does not contain a Kuratowski subgraph, then G is planar.

Proof. We prove something much stronger: if G is at least 3-connected and does not contain a Kuratowski subgraph, then G has a **convex embedding in the plane**[2] in which no three vertices lie on a line. We proceed by induction on the number of vertices of G. For $|V(G)| \leq 4$, there is only one 3-connected graph, K_4, which certainly admits such an embedding:

We now proceed to the inductive step. Take any graph G that does not contain a Kuratowski subgraph and is at least 3-connected. We know that there is some edge $e = \{x, y\}$ such that $G \cdot e$ also possesses both of these properties; by induction, then, we know that $G \cdot e$ has a convex embedding in the plane with no three vertices on a line!

We seek to show that the same property holds for G itself. To see this, take $G \cdot e$'s convex embedding. Delete the vertex v_e corresponding to e, and look at the face that contains where v_e used to be. This face must be a cycle, as if it had any branches stemming off of it then we could delete at most one vertex in such a branch to disconnect G, which would imply that $G \cdot e$ was at most 2-connected.

Notice that all of the neighbors of x and y are contained within the vertices on this cycle, because our embedding is planar. Label the neighbors of x on this cycle $x_1, \ldots x_n$, in the order given by walking around our cycle in a counterclockwise fashion. There are three possibilities for the neighbors of y:

1. There is some i such that all of the neighbors of y fall between x_i and x_{i+1} in our cycle, possibly inclusive. If this happens, we can create a convex embedding for our graph by placing x where v_e used to be and by placing y very close to x, as illustrated below:

2. Otherwise, it is possible that y shares three neighbors x_i, x_j, x_k with x. In this case, we have the following diagram:

   However, the above graph is a subdivision of K_5, and we said that G did not contain any Kuratowski subgraphs! So this case is impossible.

3. Finally, it is possible that y has two neighbors y_i, y_j that alternate with neighbors x_i, x_j of x. In this case, we have the following diagram:

[2] A **convex embedding** is a planar embedding in which all of the lines are straight lines, and moreover all of the faces are convex polygons.

The above graph is a subdivision of $K_{3,3}$, which again we know cannot exist! Therefore this case is also impossible; thus we are always in the first situation, where we were able to find a convex embedding.

□

What can we conclude from this? Well: we proved earlier that any minimally nonplanar graph without a Kuratowski subgraph must be at least 3-connected. Combining this result with the above tells us that it is impossible for a graph to be both minimally nonplanar and not contain a Kuratowski graph, as it would have to be both planar and nonplanar! Consequently, we have proven our goal for this set of talks:

Theorem. (Kuratowski's theorem) A graph is planar if and only if it does not contain a Kuratowski graph as a subgraph.