Math 137B	Professor: Padraic Bartlett

Homework 6: Sequential Dynamical Systems

Due Thursday, week 9 — *UCSB 2014*

Pick **three** of the problems in this set to solve! Solutions need justification and proof to receive full credit: i.e. it is not enough to simply draw the answer.

1. Given a SDS $[F_G, \pi]$ on a graph with n vertices with states from some set K, we can form its **phase space** as the following directed graph:

Vertices: The elements of K^n.

Edges: Draw an edge from $\vec{x}$ to $\vec{y}$ whenever $[F_G, \pi](\vec{x}) = \vec{y}$.

We say that two SDS's are **dynamically equivalent** if their resulting phase spaces are isomorphic graphs. How many dynamically inequivalent SDS's exist on the graph $G = C_4$ with the update map $parity(x_{i-1}, x_i, x_{i+1}) = x_{i-1} + x_i + x_{i+1} \mod 2$? (In other words, how many update orders on this graph+update function pairing lead to dynamically inequivalent SDS's?)

2. In class, we proved that if $\vec{x}$ is a fixed point of a SDS $[F_G, \pi]$, then it is a fixed point of $[F_G, \pi']$ for any other permutation π' of the vertices of G. What happens if we instead look at "word-SDS's," where we allow π to simply be any sequence of the vertices of G: does this theorem still hold? What if we only allow words that use each vertex at least once?

3. (This is the Cauchy-Frobenius lemma[1] ; only do this problem if you have not done it in an algebra class before.) Given a group G and a set X, we can often create a way for the group G to **act** on the set X. We define this formally as follows:

   Definition. A **group action** of a group G on a set X is a function $G \times X \to X$, typically denoted $g \cdot x$, with the following two properties:

   - **Compatibility**: for any $g, h \in G$, $g \cdot (h \cdot x) = (g \cdot h) \cdot x$. In other words, if I first act on an element of X by h and then act on it by g, this should be the same as acting on it once by the element $g \cdot h$. Note that $g \cdot h$ is calculated using the group's operation, and not our action.
   - **Identity**: $e \cdot x = x$, for any $x \in X$.

   A natural example of a group action is the group of permutations S_n acting on a collection of n objects: each permutation map sends some object to another object.

[1] Also known in the literature as the Burnside lemma, or often as "The lemma that is not Burnside's." This is because (1) Burnside did not prove this lemma, and only cited it in his textbook with a credit to Frobenius. Frobenius also was not the first to this lemma, as Cauchy knew about it forty years before Frobenius's first text mentioned it. Math history!

Given a group action, we define the **orbit** of an element $x \in X$ as the following set:

$$G(x) = \{g \cdot x \mid g \in G\}.$$

We also define the **fixed points** corresponding to an element $g \in G$ as the following set:

$$Fix(g) = \{x \in X \mid g \cdot x = x\}.$$

(a) Show that any two orbits must either be disjoint or identical: i.e. that for any $x, y \in X$, either $G(x) = G(y)$ or $G(x) \cap G(y) = \emptyset$.

(b) Show that the number of distinct orbits can be counted by the following formula:

$$\frac{1}{|G|} \sum_{g \in G} |Fix(g)|.$$

4. Find all of the maps $f : \mathbb{F}_2^3 \to \mathbb{F}_2$ that induce invertible SDS's over the graph C_n. Each of these maps have rule numbers, as defined in our discussion of cellular automata (see below.) What are those rule numbers?

5. **Cellular automata**, as mentioned in class, are closely related to sequential dynamical systems in a number of ways. Formally, we define a **cellular automaton** over $\mathbb{Z}^k$ with states in $\mathbb{F}_2$ as the following set of objects:

Graph: The integer lattice $\mathbb{Z}^k$. Sometimes, we will work with $(\mathbb{Z}/n\mathbb{Z})^k$ instead, in cases where we do not want to work with an infinite grid.

States: Each vertex has a state in $\mathbb{F}_2$. Typically, we ask that only finitely many vertices correspond to the state 1, which we think of as denoting "life" or "activity;" conversely, we regard 0 as denoting death or quiescence.

Neighborhood: The **Moore** neighborhood[2] of any vertex $\vec{x}$ consists of all vertices $\vec{x'}$ such that $|x_i - x_i'| \leq 1$, for every coördinate i. So, for a cellular automata on $\mathbb{Z}^2$, each vertex has eight vertices in its neighborhood.

Local update function: Some function $f : \mathbb{F}_2^{3^k} \to \mathbb{F}_2$, that takes in the state of a vertex and its neighbors and outputs some updated state for that vertex.

Initial state: Some element $\vec{x} \in (\mathbb{F}_2)^{\mathbb{Z}^k}$ that describes the initial state of every vertex in $\mathbb{Z}^k$.

Given a cellular automata, we can define its **global update function** as the map Φ_f that applies the local update map f to the state of every vertex simultaneously.

(a) Explain why there are 256 possible update maps for a cellular automata on $\mathbb{Z}^1$.

(b) The **Wolfram enumeration** of update maps assigns to each update map on $\mathbb{Z}^1$ a number from 0 to 255, written out in binary, as follows:

(x_{i-1}, x_i, x_{i+1})	111	110	101	100	011	010	001	000
f	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0

[2] There are, of course, many notions of neighborhood one could take. For example, you could consider as the neighborhood of any vertex all of the vertices that are distance at most one from this vertex; this is the Von Neumann notion of neighborhood. We will work with Moore neighborhoods unless otherwise specified.

For example, the AND function

(x_{i-1}, x_i, x_{i+1})	111	110	101	100	011	010	001	000
AND	1	0	0	0	0	0	0	0

is assigned the number $10000000_{\text{binary}} = 128_{\text{dec}}$. We call this number the "local rule" for its corresponding cellular automaton.

Find the local rules for NAND, NOR, and Parity.

(c) Show that the phase spaces for the cellular automata with local rules 124, 193 and 110 are all isomorphic.

6. Recall, from class, our definition of **symmetric**: we said that a function f is **symmetric** if it does not care about the order of its inputs: i.e. $f(x_1, \dots x_n) = f(x_{\pi(1)}, \dots x_{\pi(n)})$, for any permutation π. How many cellular automata over $\mathbb{Z}^1$ with states in $\mathbb{F}_2$ and a symmetric update function are dynamically inequivalent?

7. In this exercise, we show that there are at most 88 non-equivalent phase spaces for cellular automata over $\mathbb{Z}^1$ with states in $\mathbb{F}_2$. Make the following definitions:

   - $inv(x_1, \dots x_n) = (1 - x_1, \dots 1 - x_n)$,
   - $rev(x_1, \dots x_n) = (x_n, \dots x_1)$,
   - $\gamma(a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0) = (1-a_0, 1-a_1, 1-a_2, 1-a_3, 1-a_4, 1-a_5, 1-a_6, 1-a_7)$, and
   - $\delta(a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0) = (a_7, a_3, a_5, a_1, a_6, a_2, a_4, a_0)$.

   (a) Take any automata Φ_f. Prove that the function $inv \circ \Phi_f \circ inv^{-1}$ is in fact equal to the cellular automata $\Phi_{\gamma(f)}$; consequently, Φ_f and $\Phi_{\gamma(f)}$ are isomorphic.

   (b) Similarly, show that $\Phi_{\delta(f)}$ is equal to $rev \circ \Phi_f \circ rev^{-1}$, and therefore that Φ_f and $\Phi_{\delta(f)}$ are isomorphic.

   (c) Consider the set $G = \{id, \gamma, \delta, \gamma \circ \delta\}$. Show that this is a group.

   (d) The group G acts on the set R of rules for our cellular automata (i.e. given any element α of our group and any rule f, we can get a new rule $\alpha \circ f$.) You proved in (a) and (b) that this action sends a rule f to another rule that it is dynamically equivalent to. Therefore, the number of nonequivalent rules is bounded above by the number of distinct orbits of this group action. Use the Frobenius lemma to show that this quantity is bounded about by 88.

8. (Harder, or at least more tedious.) Show that there are **exactly** 88 non-equivalent phase spaces for for cellular automata over $\mathbb{Z}^1$ with states in $\mathbb{F}_2$.