Math 137B — Professor: Padraic Bartlett

Homework 5: Planar and Nonplanar Graphs

Due Thursday, week 8 — *UCSB 2014*

Pick **three** of the problems in this set to solve! Solutions need justification and proof to receive full credit: i.e. it is not enough to simply draw the answer.

1. This problem modifies Kuratowski's theorem (i.e. that graphs are planar iff they do not contain subdivided K_5's or $K_{3,3}$'s) to Wagner's theorem: a graph is planar iff it does not contain one of K_5 or $K_{3,3}$ as **minors**[1]. Perform this modification as follows:
   (a) Suppose we take a planar graph G and delete an edge or a vertex. Prove G is planar.
   (b) Suppose we take a planar graph G and contract an edge e in G to a point: i.e. that we form the graph $G \cdot e$. Prove that $G \cdot e$ is planar.
   (c) Suppose that G is a graph such that it contains one of K_5 or $K_{3,3}$ as a minor. Explain why G cannot be planar.
   (d) Suppose that G does not contain either of K_5 or $K_{3,3}$ as minors. Explain why Kuratowski's theorem says that G is planar.

2. Given a connected planar graph G, we can form the **dual** to this graph, G^*, as follows:
   - Vertices of G^*: the faces of G.
   - Edges of G^*: connect two faces F_1, F_2 if they share an edge in common.

   (a) Prove that G^* is a connected planar graph.
   (b) Show that the dual of G^*, i.e. $(G^*)^*$, is just the graph G again.
   (c) Draw the five platonic solids as planar graphs.
   (d) Find the dual of each solid.

3. A graph G is called **outerplanar** if it can be drawn in the plane in such a way that every vertex of G lies on the outer face of G.
   (a) Show that K_4 and $K_{2,3}$ are planar but not outerplanar.
   (b) Show that every outerplanar graph has a vertex of degree at most 2.

4. Take any outerplanar graph G. Form the dual G^* of G, and delete the vertex of G^* that corresponded to the "outer" face of G; call this new graph $G^\ddagger$. Show that $G^\ddagger$ is a **forest**: i.e. it is a graph that can be drawn as a collection of disjoint trees.

5. Prove that a graph G is outerplanar if and only if it contains no subgraph that is a subdivision of K_4 or $K_{2,3}$. (Hint: while you could try to recreate an outerplanar version of Kuratowski's theorem, a saner approach is to actually look for a way to modify G and simply **apply** Kuratowski's theorem.)

[1] A graph G contains another graph H as a **minor** if there is some way to contract and delete edges/vertices from G to get the graph H