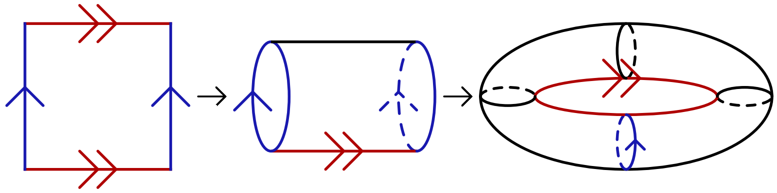
Math 137B — Professor: Padraic Bartlett

Homework 4: Toroidal Graphs

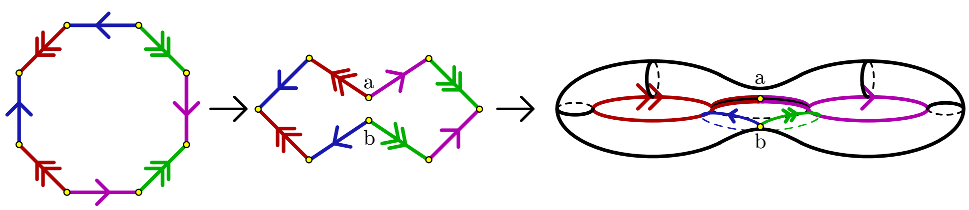
Due Thursday, week 7 — *UCSB 2014*

Pick **4** of the problems in this set to solve! Solutions need justification and proof to receive full credit: i.e. it is not enough to simply draw the answer.

1. In class, we created a way to glue together sides of a square to make a torus:



The picture below shows how to glue together sides of an octagon to create a "two-hole torus:"



   (a) Create a way to glue together the sides of a square to get a sphere.

   (b) Create a way to glue together the sides of a $4n$-gon to get a n-hole torus.

   (c) Create a way to glue together the opposite sides of a hexagon to get a torus.

2. A graph G drawn on a n-hole torus is called n-**toroidal** if it satisfies the same definition we gave in class (i.e. we can draw it on a n-hole torus so that no edges intersect and the regions bounded by edges look like open regions of $\mathbb{R}^2$.) Prove that if G is a 2-toroidal graph, then $V - E + F = -2$.

3. Generalize the above problem: show that if G is an n-toroidal graph, then $V - E + F = 2 - 2n$.

4. Prove Heawood's formula: if G is an n-toroidal graph, then

$$\chi(G) \leq \left\lfloor \frac{7 + \sqrt{1 + 48n}}{2} \right\rfloor.$$

5. Suppose that G is planar. Prove that there is a planar embedding of G in the plane where all of the edges are drawn with straight line segments.

6. (a) Show that there is no connected bipartite 3-regular planar graph of order 10.

(b) Show that for any $n \geq 4, n \neq 5$, there is a connected bipartite 3-regular planar graph of order $2n$.

7. Let G be a planar n-vertex graph with girth k (i.e. a graph that contains a k-cycle as a subgraph, but no smaller cycles as subgraphs.)

   (a) Prove that G has at most $(n-2)\frac{k}{k-2}$ edges.

   (b) Explain why this means the Petersen graph is nonplanar.

8. Find the smallest number of edges you need to delete from the Petersen graph to make it planar.

9. (a) Show that if G is a planar graph on 11 vertices, then the complement of G is nonplanar.

   (b) Find a planar graph G on 8 vertices such that its complement is planar.