Math 137B	Professor: Padraic Bartlett

Homework 2: Cayley Graphs

Due Thursday, week 4 — *UCSB 2014*

Pick **five** of the problems in this set to solve! Solutions need justification and proof to receive full credit: i.e. it is not enough to simply draw the answer.

1. Draw the Cayley graph for the quaternion group $\langle a, b \mid a^2 = b^2 = (ab)^2 \rangle$.

2. Create the Cayley graph for S_4 with generators $(1, 2, 3, 4)$ and $(1, 2)$.

3. Show that the dihedral group D_{2n} discussed in class can be expressed via the presentation $\langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle$. Find its Cayley graph.

4. Let $\mathbb{Z}_n$ denote the group given by taking the set $\mathbb{Z}/n\mathbb{Z}$ along with the addition mod n operation. Find the Cayley graph of $\mathbb{Z}_n \times \mathbb{Z}_m$ with respect to the generators $(1, 0), (0, 1)$, for any n, m.

5. For any odd n, find a group G with generating set S such that its Cayley graph is an oriented K_n. (An **oriented** K_n is just a copy of the complete graph K_n where we assign a direction to each edge. These graphs are also called **tournaments**.)

6. Let Q_n denote the graph corresponding to the n-dimensional unit cube. Find a group G with generating set S such that its Cayley graph is the unoriented graph Q_n. (By an **unoriented** graph, we are asking that whenever we have an edge (x, y) in our Cayley graph, we want to also have the reverse edge (y, x).)

7. Recall, from last quarter, the following definitions:

   Definition. Given two graphs G_1, G_2 with vertex sets V_1, V_2 and edge sets E_1, E_2, we say that a function $f : V_1 \to V_2$ is an **isomorphism** if the following two properties hold:

   - f is a bijection.
   - (x, y) is an edge in E_1 if and only if $(f(x), f(y))$ is an edge in E_2.

   An **automorphism** on a graph G is an isomorphism from that graph to itself.

   Using this definition, we say that a graph G is **vertex-transitive** if given any two vertices v_1, v_2 of G, there is an automorphism f on G such that $f(v_1) = v_2$. In essence, vertex-transitive graphs have a lot of symmetry: up to the labeling, we cannot distinguish any two vertices.

   Prove that any Cayley graph is a vertex-transitive graph.

8. Prove or disprove: there is no group has the Petersen graph as its Cayley graph.

9. Prove or disprove: there is no group has the dodecahedron graph as its Cayley graph.

10. (Applicable mostly to students with musical background; from VanWyk's class at James Madison.) Consider the following "twelve-tone group," formed by taking the twelve pitches $C, C^\sharp, \ldots B$:

	C	$C^\sharp$	D	$D^\sharp$	E	F	$F^\sharp$	G	$G^\sharp$	A	$A^\sharp$	B
C	C	$C^\sharp$	D	$D^\sharp$	E	F	$F^\sharp$	G	$G^\sharp$	A	$A^\sharp$	B
$C^\sharp$	$C^\sharp$	D	$D^\sharp$	E	F	$F^\sharp$	G	$G^\sharp$	A	$A^\sharp$	B	C
D	D	$D^\sharp$	E	F	$F^\sharp$	G	$G^\sharp$	A	$A^\sharp$	B	C	$C^\sharp$
$D^\sharp$	$D^\sharp$	E	F	$F^\sharp$	G	$G^\sharp$	A	$A^\sharp$	B	C	$C^\sharp$	D
E	E	F	$F^\sharp$	G	$G^\sharp$	A	$A^\sharp$	B	C	$C^\sharp$	D	$D^\sharp$
F	F	$F^\sharp$	G	$G^\sharp$	A	$A^\sharp$	B	C	$C^\sharp$	D	$D^\sharp$	E
$F^\sharp$	$F^\sharp$	G	$G^\sharp$	A	$A^\sharp$	B	C	$C^\sharp$	D	$D^\sharp$	E	F
G	G	$G^\sharp$	A	$A^\sharp$	B	C	$C^\sharp$	D	$D^\sharp$	E	F	$F^\sharp$
$G^\sharp$	$G^\sharp$	A	$A^\sharp$	B	C	$C^\sharp$	D	$D^\sharp$	E	F	$F^\sharp$	G
A	A	$A^\sharp$	B	C	$C^\sharp$	D	$D^\sharp$	E	F	$F^\sharp$	G	$G^\sharp$
$A^\sharp$	$A^\sharp$	B	C	$C^\sharp$	D	$D^\sharp$	E	F	$F^\sharp$	G	$G^\sharp$	A
B	B	C	$C^\sharp$	D	$D^\sharp$	E	F	$F^\sharp$	G	$G^\sharp$	A	$A^\sharp$

(a) Explain briefly why this group is isomorphic to $\mathbb{Z}_{12}$.

(b) Write out the Cayley graph given by this group, with generator F.

(c) Interpret (b): what is this structure?