Homework 4: SRG's/Probabilistic Method/Random Graphs/Flows

Due Tuesday, March 11, in class *UCSB 2014*

Homework problems need to show work and contain proofs in order to receive full credit. Simply stating an answer is only half of the problem in mathematics; you also need to include an argument that persuades your audience that your answer is correct.

This HW set should be split into two pieces when it is turned in. One part should contain **one** selected problem that you want to be carefully graded on a ten point scale! The other part should contain the other ten problems, each of which will be graded on a one-point scale (i.e. 1/.5/0 for work that is correct/mostly correct but flawed/incorrect.)

Have fun!

1. The **girth** of a graph G is the length of the shortest nontrivial cycle it contains. For example, the girth of a K_4 is 3, as it contains a triangle, the shortest nontrivial 3-cycle. Similarly, the girth of a C_n is n, as it contains a n-cycle but does not contain any shorter cycles. If a graph contains no cycles, we say that its girth is ∞.

   (a) Explain why any strongly regular graph with parameters[1] $(n, k, 0, \mu)$ has girth at least 4.

   (b) Take any $(n, k, 0, 1)$, where $n \geq 5$. Prove that it has girth 5.

2. In class, we mentioned that finding $R(3, 3)$ can be thought of as a game:

   - There are two players, Red and Blue. Their gameboard consists of six points drawn on a plane. Players alternate turns, and Red starts first.
   - On a given player's turn, they must connect two points that do not have a line drawn between them yet, with a line of their given color.
   - The game ends when a monochromatic triangle is drawn on the board, in which case that player loses.

   We proved in class that this game always ends with one player losing (i.e. no draws are possible), because $R(3, 3) = 6$. Assuming that both players play optimally, which player always wins: red or blue? Construct a strategy for one of the players that guarantees that they will win.

3. Determine the exact value of $R(3, 5)$.

4. The **distance** between any two vertices in a graph G is the length of the shortest walk connecting those vertices; the **diameter** of a graph G is the maximum of the distance function over all of the vertices in G. For example, the diameter of K_n is 1 for any n, while the diameter of C_n is $\lfloor n/2 \rfloor$.

   Show that the probability of a random graph $G_{n,1/2}$ having diameter 2 goes to 1 as n goes to infinity.

[1] Fun fact: there are only 7 known nontrivial $(n, k, 0, \mu)$'s in existence. Feel free to find an eighth in place of this HW/your final...

5. Take any subset B of n positive integers $\{b_1, \dots b_n\}$. Prove that B contains a sum-free[2] subset of size $\geq n/3$.

   Hint: pick some prime p that's larger than twice the maximum absolute value of elements in B, and look at B modulo p (in other words, look at B as a subset of $\mathbb{Z}/p\mathbb{Z}$. Because of our clever choice of p, all of the elements in B are distinct mod p (why?) Now, look at the sets $xB := \{x \cdot b : b \in B\}$ in $\mathbb{Z}/p\mathbb{Z}$. Using the probabilistic method, show that there is some value of x such that more than a third of the elements of xB lie between $p/3$ and $2p/3$. Use this to attack your problem.

6. Take the Rado graph R from Thursday, week 8's lecture. Delete finitely many vertices and edges from this graph, to get some new graph R'. Is R' isomorphic to R? If it is, prove your claim; if it is not, give a construction that explains why this fails.

7. Consider the following graph on the vertex set $\mathbb{N}$, where we draw an edge $\{x, y\}$ whenever the x-th bit of y's binary representation is 1, or the y-th bit of x's binary representation is 1. Show that this graph is isomorphic to the Rado graph.

8. Suppose that you have a $n \times n$ matrix of real numbers A. Let c_i denote the sum of all of the elements in the i-th column of A, and r_i denote the sum of all of the entries in the i-th row of A. A **rounding** of A is the act of taking each value a_{ij}, r_i, c_j and rounding these numbers either up or down to integer values. A rounding is called **successful** if in the resulting rounded matrix A_R, the row and column sums are the same things as the values we chose to round the r_i, c_j's to. We give an example of a successful and an unsuccessful rounding below:

0.6	0.8	2.7	4.1
0.3	1.9	2.7	4.9
2.3	0.4	0.4	3.1
3.2	3.1	5.8	

unsuccessful rounding $\longrightarrow$

1	1	3	5
0	2	3	5
2	0	0	3
3	3	6	

,

0.6	0.8	2.7	4.1
0.3	1.9	2.7	4.9
2.3	0.4	0.4	3.1
3.2	3.1	5.8	

successful rounding $\longrightarrow$

1	0	3	4
1	2	2	5
2	1	0	3
4	3	5	

.

   Prove, ideally using the Max-Flow-Min-Cut theorem, that every real-valued $n \times n$ matrix has a successful rounding.

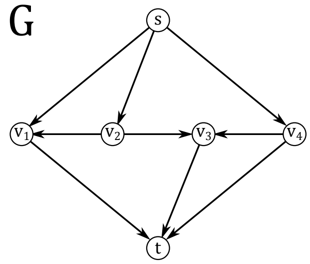
9. Use Ford-Fulkerson's Max-Flow-Min-Cut theorem to prove the following result of Hall:

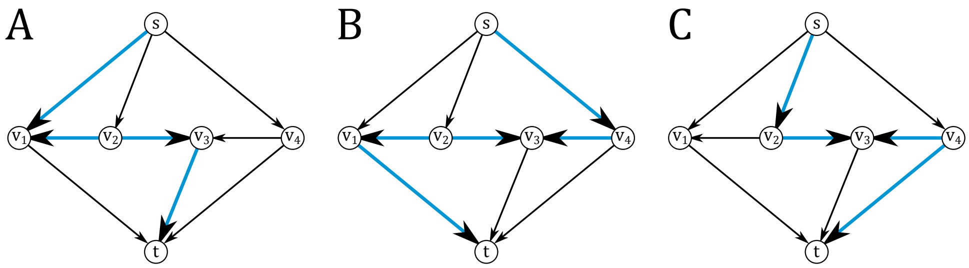
   Theorem. Let G be a bipartite graph with bipartition $V_1 \cup V_2$. Suppose that G satisfies the following property: for any collection of vertices $S \subseteq V_1$ or V_2, $N(S)$, the collection of all neighbors of elements of S, is not a smaller set than S.

   Then G has a **perfect matching**: i.e. there is a collection $M \subseteq E(G)$ of edges, such that every vertex in G is the endpoint of exactly one edge in M.

[2] A subset S of $\mathbb{R}$ is called **sum-free** if for any $a, b \in S$, $a+b$ is not in S. For example, $\{1, 3, 5\}$ is sum-free.

10. This graph is meant to illustrate the dangers of irrational flows and the Ford-Fulkerson algorithm for finding maximum flows. Consider the following graph G, which we've drawn below along with three distinct paths A, B, C:





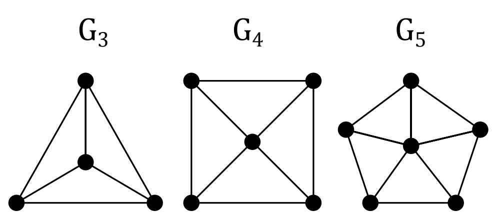
Suppose that the capacity function on this graph has $c(v_2 \to v_1) = 1, c(v_2 \to v_3) = 1, c(v_4 \to v_3) = \Phi = \frac{\sqrt{5}-1}{2}$, and the capacity of all other edges is some really large constant: say the speed of light $(3 \cdot 10^8.)$

Run Ford-Fulkerson on this graph by picking the following specific augmenting paths:

(a) At the very start, augment on the path $(s, v_2), (v_2, v_3), (v_3, t)$.
(b) Now, augment on path B.
(c) Now, augment on path C.
(d) Now, augment on path B.
(e) Now, augment on path A.
(f) Repeat, starting at step (b).

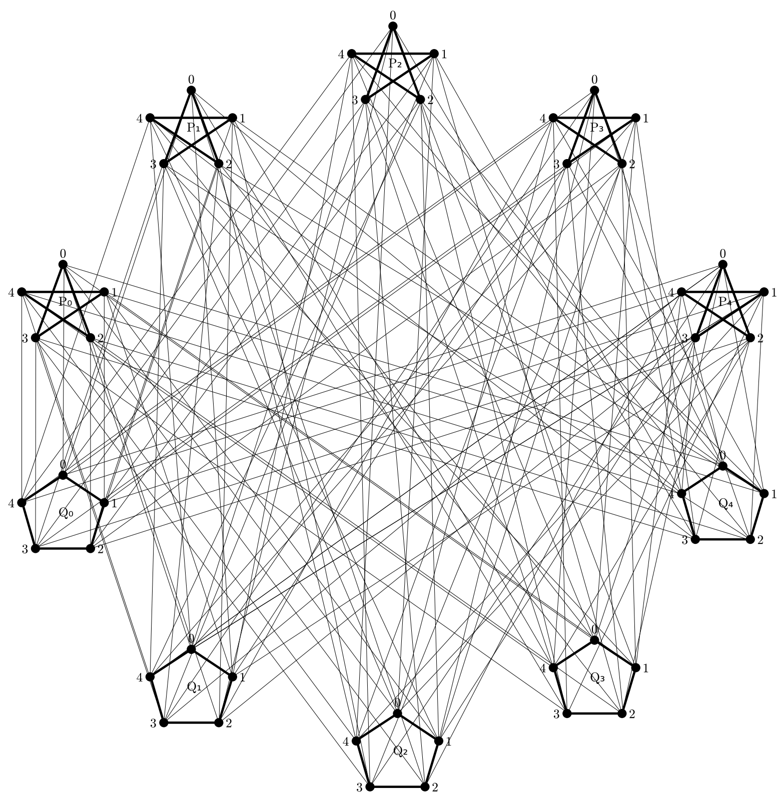
Show that all of these paths are augmenting, and thus that Ford-Fulkerson could conceivably pick these paths; show that the process above converges to a flow with value $2 + \sqrt{5}$, and finally notice that this is far far less than its maximum possible value!

11. Consider the following family G_n of graphs, formed by taking a cycle C_n and adding a new vertex with edges to every vertex in the cycle. Classify all of the values of n for which $\varphi(G_n) = 3$.



Bonus! This is worth one point. Beat me at the game mentioned in problem 2, in office hours or after class. You are allowed to choose your color. One try per student.

Bonus! This is worth 3 points. If you attempt this problem, either email me your solutions or attach them to the starred problem, so I see it. In class, we proved that there were at most four nondegenerate strongly regular graphs with parameter set $(n, k, 0, 1)$: the pentagon $(5, 2, 0, 1)$, the Petersen graph $(10, 3, 0, 1)$, the Hoffman-Singleton graph with parameter set $(50, 7, 0, 1)$, and potentially a fourth graph with parameters $(3250, 57, 0, 1)$, whose existence we have not yet verified. If you remember our constructions, we saw that all of these graphs were pretty intimately related: the Petersen graph was basically a clever way to weld two pentagons together, and the Hoffman-Singleton graph (depicted below) was a clever way to weld five Petersen graphs together:



The Hoffman-Singleton graph: Take five stars $P_0, \ldots P_4$ and five pentagons $Q_0, \ldots Q_4$. Enumerate the vertices of each pentagon and star in counterclockwise order as $0, 1, 2, 3, 4$. For every triple i, j, k, connect the vertex i in P_j to the vertex $i + jk$ in Q_k.

Your task[3], if you wish to accept it, is to try to mimic this construction for the $(3250, 57, 0, 1)$ graph. I.e. suppose you have $3250 = 5^5 + 5^3$ vertices. Is there some clever way to break these vertices up into 650 pentagons, and then glue them together such that your graph is regular with degree 57?

[3] Note that this isn't asking you to explicitly construct a $(3250, 57, 0, 1)$; rather, I'm just asking if you can stick together a ton of pentagons in a way that satisfies the 3250, 57 part of the construction.