Math 137a — Professor: Padraic Bartlett

Homework 3: Spectral Graph Theory

Due Thursday, February 20, in class — *UCSB 2014*

Homework problems need to show work and contain proofs in order to receive full credit. Simply stating an answer is only half of the problem in mathematics; you also need to include an argument that persuades your audience that your answer is correct.

THis HW set should be split into two pieces when it is turned in. One part should contain **one** selected problem that you want to be carefully graded on a ten point scale! The other part should contain the other ten problems, each of which will be graded on a one-point scale (i.e. 1/.5/0 for work that is correct/mostly correct but flawed/incorrect.)

Have fun!

1. In class, we created a formula to count the number of triangles in a graph. Create a similar formula to count the number of 4-cycles in a given graph. To make sure that your formula works, test it on C_4.

2. Let G be a finite graph, and w_p the number of closed walks of length p that exist on G. As an example, the graph C_3 contains six closed walks of length 3; one for each pair of (starting point) + (orientation.)

   Suppose that p is a prime number. Prove that w_p is divisible by p.

3. Prove or disprove the following claim: for any n and any graph G, the number of closed walks of length n in G, w_n, is divisible by n.

4. Let G be a finite graph with $n \times n$ adjacency matrix A_G and eigenvalues $\lambda_1, \dots \lambda_n$ (where each eigenvalue is listed as many times as its multiplicity indicates.)

   Prove that $\lambda_1^2 + \dots \lambda_n^2$ is a nonnegative integer. What very simple graph property does this sum correspond to?

5. Prove or disprove: there is no graph with $\sqrt{3+\sqrt{17}}$ as an eigenvalue.

6. Prove or disprove: there is no graph with $-1/2$ as an eigenvalue.

7. Let $K_{n,m}$ be the complete bipartite graph on (n, m) vertices (i.e. $K_{n,m}$ consists of two parts, one with n vertices and the other with m vertices. Every possible edge from one part to the other exists, and no edges exist within one given part.) Find the spectra of $K_{n,m}$.

8. Prove or disprove the following claim: there is some way to partition the edges of the graph K_{10} into three groups, each isomorphic to copies of the Petersen graph.

9. Find the spectra of the undirected n-cycle C_n.

10. A **Latin square** is a $n \times n$ array filled with the symbols $\{1 \dots n\}$, so that there are no repeated symbols in any row or column. Here is an example for when $n = 3$:

1	2	3
2	3	1
3	1	2

Given any such square, form its corresponding **Latin square graph** as follows: create n^2 vertices, one for each cell in the square. Connect two vertices by an edge whenever one of the three following conditions hold:

- The two cells corresponding to this vertex lie in the same row.
- The two cells corresponding to this vertex lie in the same column.
- The two cells corresponding to this vertex contain the same symbol.

Prove that this graph is regular with degree $3(n-1)$. Show that this graph has precisely three distinct eigenvalues in its spectrum. (Hint: use the same methods we used to find the Petersen graph's spectrum!)

11. Prove or disprove: if G is a graph with eigenvalues $\lambda_1, \dots \lambda_n$, then $\overline{G}$ is a graph with eigenvalues $n - \lambda_1, \dots n - \lambda_n$.