Math 137a — Professor: Padraic Bartlett

Homework 2: Colorings

Due Tuesday, January 21, in class — *UCSB 2014*

Homework problems need to show work and contain proofs in order to receive full credit. Simply stating an answer is only half of the problem in mathematics; you also need to include an argument that persuades your audience that your answer is correct.

THis HW set should be split into two pieces when it is turned in. One part should contain **one** selected problem that you want to be carefully graded on a ten point scale! The other part should contain the other ten problems, each of which will be graded on a one-point scale (i.e. 1/.5/0 for work that is correct/mostly correct but flawed/incorrect.)

Have fun! Also, there are two bonus questions. Solve either and we'll, um, get your work published in a good journal?

1. In Tuesday, week 3's class, we introduced the "greedy algorithm" for coloring vertices (described on p.120 of your text, if you've forgotten how it goes.) Using this bound, we proved that for any graph G, we had $\chi(G) \leq \Delta(G) + 1$. However, we mentioned that for some graphs, this bound was very weak, and gave some examples. However, this didn't show that running the algorithm actually **gave** a worse coloring than the theoretical best: just that it might!

   Do this: i.e. find a graph G and vertex enumeration $(v_1, \ldots v_n)$ of $V(G)$ such that using the greedy algorithm to color G's vertices uses more than $\chi(G)$ colors.

2. Show that for any graph G, there is **some** enumeration $(v_1, \ldots v_n)$ of $V(G)$ such that using the greedy algorithm to color G's vertices uses **exactly** $\chi(G)$ colors. (This is sort of the converse of problem 1: in that problem, you proved that it is **possible** for the greedy algorithm to mess up, while in this problem you're proving that it is also possible that the greedy algorithm does **not** mess up.)

3. Define the **unit distance graph**, denoted in the literature as $\mathbb{R}^2$, as follows:

   - Vertex set: the collection of all points in $\mathbb{R}^2$.
   - Edge set: connect two vertices if their corresponding points in $\mathbb{R}^2$ are distance 1 apart in the plane.

   This problem asks you to find the chromatic number of $\mathbb{R}^2$.

   (a) One easy lower bound is that you'll need at least three colors. This is because there is an equilateral triangle with side length 1 in $\mathbb{R}^2$, and therefore if we're coloring all of $\mathbb{R}^2$ we'll need to give those three points different colors, or we'll have an edge with monochromatic endpoints.
   Improve this bound by 1: i.e. prove that $\chi(\mathbb{R}^2) \geq 4$.

   (b) Find any finite upper bound on the chromatic number of $\mathbb{R}^2$: i.e. find some $n \in \mathbb{N}$ such that $\chi(\mathbb{R}^2) \leq n$.

4. On Thursday, we presented Kempe's original proof of the four-color theorem from 1879, that stood for eleven years before someone found a flaw in the proof. Find that flaw: i.e. show that Kempe's algorithm for coloring graphs will fail for some specific graph or family of graphs that you can come up with. (It may be easier to describe a family of graphs than an explicit counterexample that is miscolored by Kempe's algorithm, but either is possible and neither is much more work than the other.)

5. In class on Tuesday, we described the Mycielski process, that takes in a triangle-free graph with chromatic number k and outputs a triangle-free graph with chromatic number $k+1$. The following problem is a stronger process: it aims to take in a graph with chromatic number k that do not contain any C_3, C_4, or C_5's, and output a graph with chromatic number $k+1$ that still does not contain any C_3, C_4, or C_5's.

   We do this as follows: Let G be a k-chromatic graph with no cycles of length 5 or smaller, with vertex set $\{v_1, \ldots v_n\}$. Construct a new graph G' as follows:

   - Let T be a set of kn vertices, $\{t_1, \ldots t_{kn}\}$ with no edges between them.
   - Take $\binom{kn}{n}$ disjoint copies of G, one for every n-subset of $\{1, \ldots kn\}$ and index them by these subsets: i.e. for any subset $\{i_1, \ldots i_n\} \subseteq \{1, \ldots kn\}$, make a subgraph $G_{\{i_1,\ldots i_n\}}$.
   - Take each $G_{\{i_1,\ldots i_n\}}$, and connect the vertices of G to the corresponding vertices in T given by G's indexing subset. In other words, throw in the edges $\{v_1, t_{i_1}\}, \{v_2, t_{i_2}\}, \ldots \{v_n, t_{i_n}\}$ to our graph made by the the G's and the set T.

   Show that this graph still has girth ≥ 6, as well as chromatic number $> \chi(G)$.

6. A graph $G = (V, E)$ is called k**-critical** if $\chi(G) = k$, while for any $v \in V$, $\chi(G - \{v\}) \leq k - 1$. (Note: the graph $G - \{v\}$ is precisely the graph acquired by taking G and deleting the vertex v along with all edges incident to v.) In other words, G has chromatic number k, but G with any vertex deleted has chromatic number strictly less than k.

   (a) Prove that any graph G with $\chi(G) = k$ has a k-critical subgraph.

   (b) Prove that any k-critical graph is connected.

   (c) Prove that if G is k-critical, then $G - \{v\}$ has chromatic number $k - 1$.

7. Prove that if G is k-critical, then the degree of any vertex in G is at least $k - 1$.

8. Classify the set of all 3-critical graphs.

9. A graph is called **double-critical** if $\chi(G \setminus \{x, y\}) = \chi(G) - 2$, for any pair of **adjacent** vertices in G. Suppose that G is a double-critical graph on $n \leq 4$ vertices. Prove that G is the complete graph K_n.

10. Given a collection $I\{I_1, \ldots I_n\}$ of intervals on the real line, define the **interval graph** G_I on the vertex set $\{v_1, \ldots v_n\}$ by drawing an edge $\{v_i, v_j\}$ if and only if $I_i \cap I_j \neq \emptyset$. Show that any interval graph is perfect: i.e. that for any induced subgraph H of G, we have $\chi(H) = \omega(H)$.

11. A **partially ordered set** $P = (X, <)$ is a collection of vertices $\{x_1, \ldots x_n\}$, along with a relation $<$ that satisfies the following two properties:

    - **Antisymmetry**: if $x < y$, we do not have $y < x$.
    - **Transitivity**: if $x < y$ and $y < z$, we have $x < z$.

    Given a partially ordered set $P = (X, <)$, we can construct the **comparability graph** G_P corresponding to this set, by having $V(G_P) = X$, and $E(G_P) = \{\{x, y\} : x < y$ or $y < x\}$. Show that every compararbility graph is perfect.

Bonus! Determine $\chi(\mathbb{R}^2)$, for $\mathbb{R}^2$ the **unit distance plane graph** defined earlier.

Bonus! Show that if G is a double-critical graph on n vertices, then G is the complete graph K_n.