Math 137a — Professor: Padraic Bartlett

Homework 1: Basic Graph Properties, Trees, and More

Due Tuesday, January 21, in class — *UCSB 2014*

Homework problems need to show work and contain proofs in order to receive full credit. Simply stating an answer is only half of the problem in mathematics; you also need to include an argument that persuades your audience that your answer is correct. Pick one problem to be graded in a detailed fashion, as described in the syllabus!

Also: have fun!

1. A sequence $d_1 \geq d_2 \geq \ldots d_n$ of nonnegative integers is called **graphic** if and only if there is a graph G on n vertices such that $\deg(v_i) = d_i$, for every $v_i \in V(G)$.

   Determine whether any of the following sequences are graphic:

   - $5, 3, 3, 2, 2, 2$.
   - $6, 2, 2, 2$.
   - $3, 2, 2, 2, 1, 1, 1$
   - $3, 3, 3, 3, 3, 3, 3, 3, 3, 3$
   - $\underbrace{n, n, n \ldots n}_{n+1 \text{ times}}$.

2. Prove or disprove: every graph contains two vertices of equal degree.

3. Suppose that G is a graph and k is a positive integer ≥ 2 such that $\deg(v) \geq k$, for every $v \in V(G)$. Prove or disprove the following claim: G contains a cycle of length $k + 1$.

4. Create four trees, one with four vertices, one with five vertices, one with six vertices, and one with seven vertices. Run the Prüfer algorithm on each of these trees to turn them into sequences. Run the inverse algorithm we described in class to turn them back into trees. Are they the same tree?

5. Suppose that a graph G has m edges. A m-**vertex-labeling** of this graph is a way to assign the numbers $\{0, \ldots m\}$ to the vertices of G. Given any such vertex labeling, we get an **induced** m-labeling of the edges of this graph as follows: if an edge $e = \{x, y\}$ has x labeled l_x, y labeled l_y, we can label the edge $e = |l_x - l_y|$.

   We call a m-vertex-labeling **graceful**[1] if the following two properties hold:

   - No two vertices have the same label.
   - No two edges have the same label in the induced labeling described above.

   Prove that all paths P_n have graceful labelings.

[1] An open problem in mathematics: do all trees admit graceful labelings?

6. Show that K_n has a graceful labeling if and only if $n \leq 4$.
7. Enumerate all of the distinct trees on 6 vertices up to isomorphism. (I.e. there are lots of trees on 6 vertices. If we regard isomorphic trees as the "same," however, then there are many less! Find all of these trees.)
8. Prove or disprove the following: if G is a graph where every vertex has degree at least k, and T is a tree with k edges, then G contains T as a subgraph.
9. Given a graph G, we say that a tree T is a **spanning tree** of G if

   - T is a tree,
   - T is a subgraph of G, and
   - every vertex of G is in T.

   Prove that every connected graph contains a spanning tree.
10. Show that a graph is bipartite if and only if it does not contain an odd cycle.
11. Given a graph $G = (V, E)$, define the **complement** of G to be a graph $\overline{G}$ on the same vertices as G, such that $\{x, y\}$ is an edge in $\overline{G}$ whenever it is not an edge in G, and vice-versa.

    (a) Draw three graphs on (five/six/seven) vertices, and their complements.

    (b) Prove or disprove: for any graph G, either G or $\overline{G}$ is connected.