Math 137a — Professor: Padraic Bartlett

Final!

Due Friday, March 21, by 1pm — *UCSB 2014*

This test consists of ten problems. Roughly speaking, there is one problem for each week of the class. These problems will be graded as follows:

10 points: Your performance on those ten problems, with one point alloted for each problem.

10 points: At some point in time between now and Friday, March 21, you should schedule a 30-minute block of time to meet with me. I will ask you about a few problems that I will randomly choose from the set, and you will explain your reasoning to me for how to solve those problems. I may also ask you about similar problems to those on the set, the definitions you are using, and other such things. Points for this section will be awarded on the following scale:

- 10 points: You correctly answer all of the questions put to you, and demonstrate a clear mastery/knowlege of graph theory.
- $8-9$ points: You make some minor mistakes in your presentation, but can (with a bit of assistance) prove most of the questions I ask you.
- $6-7$ points: You have some serious flaws in some of your answers, but can answer some of the questions I ask you. It is clear that you understand the basic definitions and have worked on the problems on this set, but your work is missing some key pieces.
- $4-5$ points: You have made several key errors, and do not know some of the more fundamental definitions and concepts we've worked on in this class. While you may get a few things right, your overall performance is poor.
- $1-3$ points: You meet with me but do poorly and are unable to come up with coherent answers to the questions asked. You do not know most of the basic concepts we have discussed in class, and fundamentally do not seem prepared for further study in graph theory.
- 0 points: You do not meet with me.

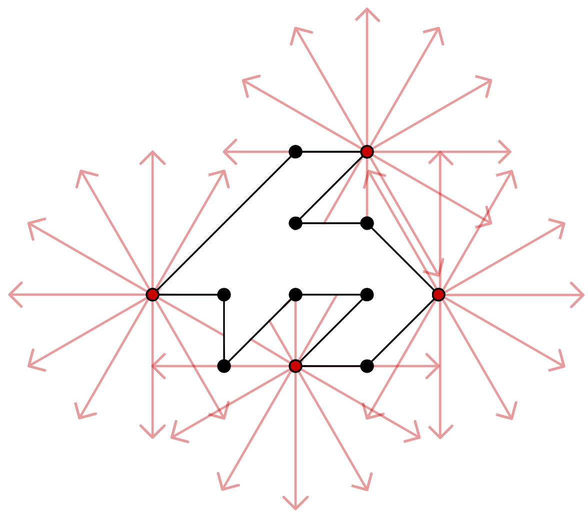
Resources allowed: You may collaborate with others. If you do so, you should cite the other person on the problems you have collaborated on, and furthermore write up your solutions separately and in your own words. So, for example, I should not look at two solutions and see word-for-word identical proofs. Solutions that egregiously violate this policy will not receive credit.

You may also consult textbooks and/or electronic resources for information. Again, if you do this, you should cite the sources you have used and write up your solutions separately and in your own words. Solutions that egregiously violate this policy will also not receive credit.

Good luck and have fun!

1. Let A be a collection of n points in $\mathbb{R}^2$ with the following property: given any two points $P, Q \in A$, the distance between P and Q is at most 1. Show that there are no more than n pairs of points that are exactly distance 1 apart.

2. Take a polygon P with n sides. Consider the following task: we want to station observers at the vertices of P, such that they can guard the entire "outside" of P. In this situation, we assume that the guards cannot "see through" P's walls, and can only look out from their positions. For example, here is a polygon P being guarded by four guards:



A polygon with observers guarding its outside. Observers are denoted by red vertices; sample sight lines are drawn in pale red.

   (a) Suppose that P is an arbitrary polygon with n vertices. What is the maximum number of guards needed to guard P?

   (b) Again, suppose that P is an arbitrary polygon with n vertices. What is the minimum number of guards needed to guard P?

3. Take the graph K_7. Color each of the edges of K_7 either green or blue. We know, by Ramsey's theorem, that there is at least one triangle in this graph with edges all the same color, because $R(3,3) = 6 < 7$. Prove that there are actually at least **four** triangles in this graph such that each triangle's edges are all the same color.

4. Take a bipartite graph G. **Without using our theorem** that a graph is perfect if and only if its complement is perfect, prove that $\chi(\overline{G}) = \omega(\overline{G})$. (Hint: While there are many proofs, one of my favorites involves applying problem 9 from this final, along with the observation that in any coloring of $\overline{G}$'s vertices, no single color is associated to more than two vertices (why?) There are other methods if you prefer, however.)

5. Suppose a graph G on n vertices contains $n^2/4$ edges. Let λ_1 denote the eigenvalue of A_G with largest magnitude. Prove that $|\lambda_1| \geq n/2$. (Note: an earlier version of this problem told you some information about the number of 4-cycles. I think this was a

red herring, in that you definitely don't need it and I don't think there's an obvious way to use it. If you want an upper bound, however, it's great! But upper bounds aren't what I actually asked for [rather, they're what I thought I asked for.])

6. Find all of the sets of parameter $(10, k, \lambda, \mu)$ such that there is a strongly regular graph with those parameters. Explain why your collection of sets is complete.

7. On an earlier quiz, we defined the **tri-color Ramsey number** $R(k, k, k)$ to be the smallest natural number n such that in any (red, green, blue)-coloring of K_n, there is a monochromatic K_k of some color. For any $k \geq 3$, prove that

$$R(k, k, k) \geq \lfloor 3^{k/2} \rfloor.$$

8. Consider the random graph process $G_{n,1/2}$, that creates a random graph by taking each pair $\{v_1, v_2\}$ of distinct vertices and randomly either drawing in an edge with probability $1/2$. Let $N(G, C_4)$ denote the number of distinct subgraphs of G that are 4-cycles, as defined in problem 5.

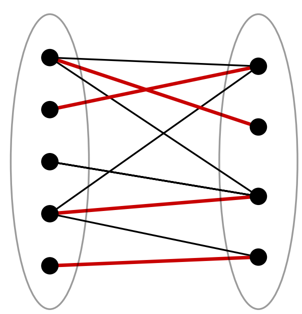
   Prove that the expected number of 4-cycles in a random graph $G_{n,1/2}$ is

$$\frac{1}{2k} \cdot \frac{n!}{(n-k)!} \cdot \frac{1}{2^k},$$

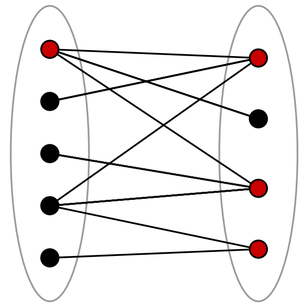
   where $k = 4$. In other words, show that if you take a random graph on n vertices, you'd expect to see about this many distinct 4-cycles.

9. Let $G = (V_1, V_2, E)$ be a bipartite graph. Consider the following two definitions:

   - A **pairing** of G's vertices is some subset of the edges of G, such that no vertex shows up in more than one edge. A sample pairing is illustrated below:

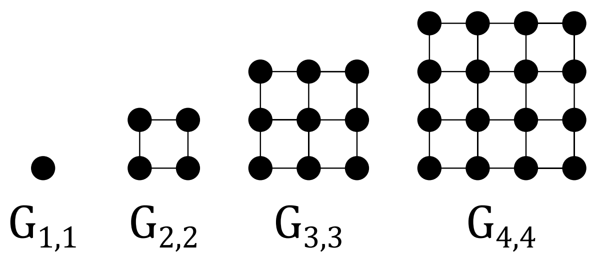


   - An **observing set** of G's vertices is some subset of the vertices of G, such that every edge of G is incident to at least one vertex in the observing set. A sample observing set is illustrated below:



Notice that in the above graph, the size of the largest matching possible (namely, 4) was the size of the smallest vertex cover possible (again, 4.) Using the Max-Flow-Min-Cut theorem, prove that this is true for any bipartite graph G: i.e. show that if G is a bipartite graph, then the size of G's largest pairing is the size of its smallest observing set.

10. For any n, let $G_{n,n}$ denote the $n \times n$ **grid graph** given by taking the subset of the integer lattice corresponding to the values $\{1, \ldots n\} \times \{1, \ldots n\}$. We draw a few examples below:



For every n, determine $\varphi(G_{n,n})$; i.e. the smallest value of k such that $G_{n,n}$ admits a k-flow.