Math 108B	Professor: Padraic Bartlett

Practice Midterm!

Actual midterm time: 9-10am, Friday, Jan. 31, Psych 1902. *UCSB 2014*

This test is out of **20** points. It is split into **two** main sections, with an additional bonus section at the end:

10 points: A **calculational** section. In this section, you are asked to calculate a series of objects related to what we've been studying in class, in a single multi-step problem.

10 points: A **proof**. In this section, you will choose **one** problem out of the two presented, and **prove** the claim it makes. If you attempt more than one problem, the first problem encountered in the blue book will be graded.

+2 points: A **bonus** question! This problem is markedly harder than the other problems on the test.

Resources allowed: up to **two** hand-written sheets of paper, in your own handwriting. Also, arbitrary amounts of blank paper, for scratch work. No textbooks or other notes are allowed. No calculators (they won't be helpful.) When proving results, you are allowed to use anything that we've proven in class or in the HW without proof. Other things, like results from LADR that we haven't proven yet or results from other books/classes, must be proven if you want to use them. It bears noting that every problem on this test can be solved using only things you have learned in this class and homework, and outside resources are borderline-useless.

All work must be written in a blue book! As with the midterm, work not in a blue book will be used for decorative origami and in particular not graded.

Good luck and have fun!

1 Calculational Problem

In order to receive points in this section, you must show your work. Solutions that simply state an answer will receive low marks.

1. Consider the matrix

$$A = \begin{bmatrix} 21 & 0 & 6 \\ 0 & 15 & -6 \\ 6 & -6 & 18 \end{bmatrix}.$$

   (a) This matrix has three eigenvalues, $\lambda_1 = 9, \lambda_2 = 18$, and $\lambda_3 = 27$. For each eigenvalue $\lambda_1, \lambda_2, \lambda_3$, find a corresponding eigenvector $\vec{v_1}, \vec{v_2}, \vec{v_3}$.

   (b) Rescale each eigenvector $\vec{v_1}, \vec{v_2}, \vec{v_3}$ so that it has unit length.

   (c) Show that these three eigenvectors form an orthonormal basis for $\mathbb{R}^3$.

   (d) Rewrite the matrix A in the basis given by this orthonormal set of eigenvectors.

2 Proof-Based Problems

Choose **one** of the two problems below. Answering more than one will result in only your first problem being graded.

1. (a) Take a $n \times n$ matrix Q. Prove that $Q^T \cdot Q^T = (Q^2)^T$, by comparing the entries of the left-hand-side matrix to the right-hand-side matrix.

   (b) Suppose that Q is an $n \times n$ matrix such that $Q^T \cdot Q = Id$, the identity matrix. Prove that Q is an **orthogonal** matrix: i.e. a matrix whose columns form an orthonormal basis for F^n (where F is either $\mathbb{R}$ or $\mathbb{C}$.) (Note: in class, we proved the converse of this statement: i.e. we showed that if Q is an orthogonal matrix, then $Q^T \cdot Q$ is the identity.)

   (c) Prove that if Q is an $n \times n$ orthogonal matrix, then Q^k is an $n \times n$ orthogonal matrix.

2. Create a 4×4 matrix A with the following properties:

   - No entry of A is 0.
   - A has 1,2,3 and 4 as eigenvalues.

3 Bonus Problem

1. Suppose that a $n \times n$ matrix A can be written both as $Q_1 \cdot R_1$ and $Q_2 \cdot R_2$, where Q_1, Q_2 are orthogonal matrices and R_1, R_2 are invertible upper-triangular matrices. Suppose further that the entries on the diagonal of R_1, R_2 are all positive. Prove that $Q_1 = Q_2$ and $R_1 = R_2$.