Math 108B	Professor: Padraic Bartlett

Practice Final!

Actual final time: 8-11am, Wednesday, March 19. *UCSB 2014*

This test is out of **20** points. It is split into **two** main sections, with an additional bonus section at the end:

20 points: A **calculational** section. In this section, you are asked to calculate a series of objects related to what we've been studying in class. Do **all** of the problems in this section.

20 points: A **proof**. In this section, you will choose **two** problems out of the three presented, and **prove** the claim it makes. If you attempt more than one problem, the first problem encountered in the blue book will be graded. Claims without proof will lose points when graded, so make sure to justify your answers.

+5 points: A **bonus** question! This problem is fun/hard.

Resources allowed: up to **four** hand-written sheets of paper, in your own handwriting. Also, arbitrary amounts of blank paper, for scratch work. No textbooks or other notes are allowed. No calculators (they won't be helpful.) When proving results, you are allowed to use anything that we've proven in class or in the HW without proof. Other things must be proven if you want to use them. It bears noting that every problem on this test can be solved using only things you have learned in this class and homework, and outside resources are borderline-useless.

All work must be written in a blue book! As with the midterm, work not in a blue book will be used for decorative origami and in particular not graded.

Good luck and have fun!

1 Calculational Problems

Do **all** of the problems here. Justify your calculations: i.e. show the work that you used to derive your answers.

1. Consider the matrix

$$A = \begin{bmatrix} 13 & -4 & 2 \\ 2 & 7 & 1 \\ -4 & 4 & 7 \end{bmatrix}$$

   Find its Schur decomposition.

2. Consider the matrix

$$B = \begin{bmatrix} 11 & 1 & 1 & 1 \\ 1 & 11 & 1 & 1 \\ 1 & 1 & 11 & 1 \\ 1 & 1 & 1 & 11 \end{bmatrix}$$

   Find a diagonal matrix D that B is similar to.

3. Consider the matrix

$$C = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

   Find a block-diagonal matrix F whose blocks are Jordan blocks that C is similar to.

4. Take the matrix

$$G = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

   Find elementary matrices E_1, E_2, E_3 such that $E_1(E_2(E_3 G E_3^{-1})E_2^{-1})E_1^{-1}$ is a matrix in Jordan normal form.

2 Proof-Based Problems

Choose **two** of the three problems below. If you answer three problems, we will grade the first two that appear in your solutions.

1. Suppose that A is a $n \times n$ matrix with the following two properties:

   - A^n is the all-zeroes matrix.
   - There is exactly one nonzero vector $\vec{v}$, up to scalar multiples, that is an eigenvector of A. (In other words, the only eigenvectors for A are vectors of the form $c \cdot \vec{v}$.)

   Find the Jordan normal form of A.

2. Suppose that A is a real-valued symmetric $n \times n$ matrix with the following two properties:

   - All of A's entries are either 0 or 1.
   - The all-1's vector is an eigenvector of A, with eigenvalue 10.

   (a) How many 1's are there in each row of A?

   (b) Suppose that $\lambda \neq 10$ is another eigenvalue of A. Prove that $\lambda \leq 10$.

3. Suppose that P is a $n \times n$ permutation matrix. Show that the determinant of P^2 is 1.

3 Bonus Problem

1. A disgruntled electrical engineer, on his last day of work, decided to rewire all of the connections between the light switches at his job. Now, each switch is connected to several different lights, and flicking that switch changes the states of all of those lights; if they are on, they become off, and vice-versa.

   After some experimentation, you have discovered the following fact: given any set of lights in the building, there is a switch that is connected to an odd number of lights in that set (as well as possibly other lights not in that set.)

   Prove that there is some way to turn off all of the lights.