Math 108B — Professor: Padraic Bartlett

# Homework 7: Similar/Elementary/Permutation Matrices

*Due Wednesday, March 5, by 1:30pm* — *UCSB 2014*

Homework problems need to show work and contain proofs in order to receive full credit. Simply stating an answer is only half of the problem in mathematics; you also need to include an argument that persuades your audience that your answer is correct! If you have any questions, feel free to contact either Yihan or I via email or office hours. Have fun!

1. Take the matrix
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}.$$
Find some sequence of elementary matrices $E_1, E_2, E_3$ such that
$$(E_1 \cdot E_2 \cdot E_3) \cdot A \cdot (E_3^{-1} \cdot E_2^{-1} \cdot E_1^{-1}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$
In other words, show that $A$ is similar to a matrix made out of just its diagonal entries!

2. Suppose that $P_{\sigma_1}, P_{\sigma_2}$ are a pair of $n \times n$ permutation matrices. Prove that $P_{\sigma_1} \cdot P_{\sigma_2}$ is also a permutation matrix.

3. Suppose that $P_\sigma$ is a $n \times n$ permutation matrix. Show that there is some value of $k$ such that $(P_\sigma)^k = I$.

4. Show that if $P_\sigma$ is a permutation matrix, it has 1 as an eigenvalue. Also, prove that if $P_\sigma$ is a permutation matrix, it does not have any eigenvalues $\lambda$ with $|\lambda| \neq 1$.

5. (a) Let $P_\sigma$ be the following permutation matrix:
$$P_\sigma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$
Find three elementary matrices $E_1, E_2, E_3$ of the form $E_{\text{switch entry } i \text{ and entry } j}$, such that $E_1 \cdot E_2 \cdot E_3 = P_\sigma$.

   (b) In general, suppose that $P_\sigma$ is a $n \times n$ permutation matrix of the form
$$P_\sigma = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}.$$
Find $n-1$ elementary matrices $E_1, E_2, \dots E_{n-1}$ of the form $E_{\text{switch entry } i \text{ and entry } j}$, such that $E_1 \cdot E_2 \cdot \ldots \cdot E_{n-1} = P_\sigma$.