Math 108B — Professor: Padraic Bartlett

Homework 5: Inner Products and Real Symmetric Matrices

Due Wednesday, February 19, by 1:30pm — *UCSB 2014*

Homework problems need to show work and contain proofs in order to receive full credit. Simply stating an answer is only half of the problem in mathematics; you also need to include an argument that persuades your audience that your answer is correct! If you have any questions, feel free to contact either Yihan or I via email or office hours. Have fun!

1. Given a matrix A, we say that B is a **cubic root** of A if $B^3 = A$.

   (a) Suppose that A is a real-valued symmetric $n \times n$ matrix. Prove that A has a cubic root.

   (b) Is this true for nonsymmetric matrices? Either find a counterexample, or prove that this is true for all matrices.

2. Suppose that A is a real-valued 2×2 matrix. For any two vectors $\vec{v}, \vec{w} \in \mathbb{R}^2$, define the "$A$-product" of $\vec{v}$ and $\vec{w}$ as follows:

$$\langle \vec{v}, \vec{w} \rangle_A = (A\vec{v}) \cdot \vec{w}.$$

   On HW#3, you studied this object for a specific value of A.

   (a) Show that for $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, this A-product is an inner product, as defined in class.

   (b) In general, suppose A is a real-valued symmetric matrix. Is this always an inner product? If so, prove it; if not, create a counterexample, and come up with conditions on A that will insure that this A-product is an inner product.

3. Suppose that A, B are a pair of complex-valued $n \times n$ unitary matrices. Prove that AB is a unitary matrix.

4. The **trace** of a $n \times n$ matrix A, denoted $\operatorname{tr}(A)$, is the sum of the entries on its diagonal. For example, the trace of $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ is $1 + 5 = 6$.

   Prove that for any two $n \times n$ matrices A, B, we have $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.