Math 108a — Professor: Padraic Bartlett

Homework 1: Motivation: Eigenthings and Orthogonality

Due Monday, January 13, in class — *UCSB 2014*

Homework problems need to show work and contain proofs in order to receive full credit. Simply stating an answer is only half of the problem in mathematics; you also need to include an argument that persuades your audience that your answer is correct! As always, if you have any questions, feel free to contact either Yihan or I via email or office hours. Have fun!

1. (a) In class, we found a 2×2 matrix with no real-valued eigenvalues. Find a 4×4 matrix with no real-valued eigenvalues. Prove that your claim is correct.

   (b) In class, we found a $n \times n$ matrix with only one eigenvalue, such that the dimension of the subspace formed by all eigenvectors for that eigenvalue was one-dimensional. Find a 4×4 matrix with exactly one eigenvalue λ, such that the dimension of the subspace formed by the collection of all eigenvectors for that eigenvalue is **two**-dimensional. Prove that your claim is correct.

2. Consider the Pell sequence $\{p_i\}_{i=1}^{\infty}$, defined recursively as follows:

   - $p_0 = 0$.
   - $p_1 = 1$.
   - $p_n = 2p_{n-1} + p_{n-2}$.

   The first ten Pell numbers are listed here:

   $$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, \ldots$$

   (a) Find a matrix $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

   $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} p_n \\ p_{n-1} \end{bmatrix} = \begin{bmatrix} p_{n+1} \\ p_n \end{bmatrix}.$$

   (b) Find the eigenvalues and corresponding eigenvectors of P.

   (c) Let $\vec{v}$ be an eigenvector of P corresponding to some eigenvalue λ, and $\vec{w}$ be some other eigenvector of P corresponding to an eigenvalue $\delta \neq \lambda$. Prove that $\vec{v}$ and $\vec{w}$ are orthogonal.

   (d) Use all of this information along with the methods we discussed in class to find p_{50}.

   (In all of your work above, prove that your claims are correct.)

3. Consider the following "miniature" model for the internet, made of six webpages:

- Pets.com: links to boo, Webvan, and Kozmo.
- boo.com: links to Webvan, Kozmo and eToys.
- Webvan.com: links to Kozmo, eToys, and DigiScents.
- Kozmo.com: links to eToys, DigiScents, and Pets.com.
- eToys.com: links to DigiScents, Pets, and boo.
- DigiScents.com: links to Pets, boo, and Webvan.

(a) Draw a diagram with six bubbles, one for each webpage, and arrows between these bubbles representing the links above.

(b) Turn this model of the internet into a 6×6 "link-matrix" A, as done in the notes.

(c) Use this link matrix to find the "importance vector" (i.e. eigenvector corresponding to the eigenvalue 1) for this matrix A, showing your work and justifying your answer. Interpret your results: what do they say about the relative importance of these six websites?