Math 108a — Professor: Padraic Bartlett

Practice Final!

Wednesday, Dec. 11, 8-11am, Phelps 3505 — *UCSB 2013*

This test is out of **45** points. It is split into **five** sections:

10 points: A **true-false** section. There will be **ten** problems here; for each, choose either T or F, and record your answer in your bluebook. No justification is needed in this part.

15 points: A **basic proofs** section. There will be **six** problems here; they will look something like the quiz problems we've seen on quizzes 5-8. You should pick **three** of these problems. For each, you should write a **proof** of this problem! Most of your points here will be based on your proof, not on your answer, so please make sure to prove your claims! If you attempt more than three, only your first three will be graded.

20 points: A **conceptual proofs** section. There will be **four** problems here; they will look something like problems from the HW. You should pick **two** of these problems. For each, you should answer the problem, and **prove** all of your claims! Again, most of your points here will be based on your proofs, not just your results.

+5 points: A **bonus** question! This problem is markedly harder than the other problems on the test. Only attempt this after you've completed absolutely everything else, and are completely sure and confident in your other answers, because seriously these are the hardest five points to earn on the test. If you attempt more than two, only your first two problems will be graded.

Resources allowed: up to **four** hand-written sheets of paper, in your own handwriting. Also, arbitrary amounts of blank paper, for scratch work. No textbooks or other notes are allowed. When proving results, you are allowed to use anything that we've proven in class or in the HW without proof. Other things, like results from LADR that we haven't proven yet or results from other books/classes, must be proven if you want to use them. It bears noting that every problem on this test can be solved using only things you have learned in this class and homework, and outside resources are borderline-useless. Also: **small snacks and water**! They're useful, great, and allowed.

Recommendations: Try to manage your time! You have a lot of it, but it's still possible to get stuck on one problem for too long. In particular, try to **not get stuck** on any true-false problem for longer than five minutes, any basic proof problem for longer than twenty, or any conceptual proof problem for longer than 30-40 minutes. It is quite likely that you will get answers to many things faster than this! But be careful about getting "stuck."

All work must be written in a blue book! As with the midterm, work not in a blue book will be used for decorative origami and in particular not graded.

Good luck and have fun!

1 True-False

For each statement below, label it as either true or false. No work is needed or relevant to your grade in this section; all I want is your T's or F's. Record your work in your blue book.

1. For any vectors $\vec{v}, \vec{w}$, the vectors $\text{orth}(\vec{v} \text{ onto } \vec{w})$ and $\vec{v}$ are orthogonal.

2. There is a linear map with domain $\mathbb{R}^6$, a two-dimensional image, and a four-dimensional nullspace.

3. n-linearity is the following property: for any two matrices A, B, we have $\det(A+B) = \det(A) + \det(B)$.

4. The determinant of any 2×4 matrix A is zero.

5. There is an elementary matrix E such that $E \cdot A$ is the matrix A with every entry multiplied by 2.

6. If T is a linear map from $\mathbb{R}^2 \to \mathbb{R}^3$, its corresponding matrix has two rows and three columns.

7. The parallelepiped spanned by the three vectors $\{(1, 1, 0), (1, 2, 0), (2, 1, 0)\}$ has positive volume.

8. There are matrices A, B such that $A \cdot B \neq B \cdot A$.

9. For any two $n \times n$ matrices A, B, $\det^+(AB) = \det^+(B) - \det^+(A)$.

10. The set $\{(x, y, z) \mid x + y + z = 0\}$ is a subspace of $\mathbb{R}^3$.

2 Basic Proofs

There are **six** basic proof problems below. Choose **three** of them, answer them, and include a **proof** of your answer! If you attempt more than three, only your first three problems will be graded.

1. Decompose the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 4 & 4 \\ 3 & 2 & 3 \end{bmatrix}$$

   into elementary matrices, and justify your answer (i.e. show the work you used to get this decomposition.) Use this decomposition to calculate $\det(A)$.

2. Find the volume of the paralleletope spanned by the three vectors

$$\{(2,1,0),(1,2,0),(1,1,1)\}.$$

3. Suppose that $T : \mathbb{R}^{12} \to \mathbb{R}^{11}$ is a linear map with nullspace given by the set

$$\text{null}(T) = \{(x_1, \ldots x_{12}) \mid x_1 + x_2 + x_3 = 0\}.$$

   Prove that T is not a surjection.

4. Consider the linear map $T : \mathbb{R}^n \to \mathbb{R}^{2n}$, given by the map

$$T(x_1, x_2, \ldots x_n) = (x_1, x_1, x_2, x_2, \ldots x_n, x_n).$$

   For example, $T(1,2,3,4)$ is $(1,1,2,2,3,3,4,4)$.

   By looking at where T sends the basis elements $\{\vec{e_1}, \ldots \vec{e_n}\}$, write T as a matrix. Justify your answer.

5. Consider the matrix

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

   Is M a surjection? Prove your claim.

6. Find the positive determinant of the following matrix:

$$\begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

   Prove that your claim is correct.

3 Conceptual Proofs

There are **four** conceptual proof problems below. Choose **two** of them, answer them, and include a **proof** of your answer! If you attempt more than two, only your first two problems will be graded.

1. Consider the function orth($\vec{v}$ onto $(1, 1, \dots 1)$). This is a map that takes in a vector $\vec{v}$ in $\mathbb{R}^n$ and outputs another vector in $\mathbb{R}^n$, that we have used before in class.

   (a) Create the $n \times n$ matrix R that corresponds to this map. Justify your proposed matrix; i.e. show the work that proves that this matrix does correspond to the map orth($\vec{v}$ onto $(1, 1, \dots 1)$) .

   (b) What is the positive determinant of R? Justify your answer.

2. Suppose that A is an $n \times n$ matrix such that A^3 is the all-zeroes matrix, i.e. the $n \times n$ matrix in which every entry is 0.

   (a) Think of A as a linear map from $\mathbb{R}^n \to \mathbb{R}^n$. Prove that the range of A cannot be equal to $\mathbb{R}^n$.

   (b) Create an example of such a matrix A, such that A and A^2 are not themselves all-zeroes matrices. Make sure to show that A has the properties requested.

3. Take any $n \times n$ matrix M.

   (a) Take any $k > 0 \in \mathbb{N}$, and think of M, M^k as a pair of linear maps from $\mathbb{R}^n \to \mathbb{R}^n$. Prove that

   $$\text{nullspc}(M^k) \supseteq \text{nullspc}(M).$$

   (b) Suppose that $\det^+(M) = 0$. Prove that $\det^+(M^k) = 0$ as well.

4. In class, we proved that any $n \times n$ matrix A can be written as the product $E_1 \cdot \ldots \cdot E_n$ of elementary matrices, i.e. matrices of the form

   - $E_{\text{multiply entry k by } \lambda}$, for any λ and k,
   - $E_{\text{switch entry } k \text{ and entry } l}$, for any k and l, and
   - $E_{\text{add } \lambda \text{ copies of entry } k \text{ to entry } l}$, for any values of λ, k, and l.

   Suppose that we don't get to use all of these matrices. In particular, suppose that you are restricted to the set of matrices

   - $E_{\text{multiply entry k by } \lambda}$, for any λ and k,
   - $E_{\text{switch entry } k \text{ and entry } l}$, for any k and l, and
   - $E_{\text{add } \lambda \text{ copies of entry } k \text{ to entry } l}$ with any k, l, but only for $\lambda = 1$!

   Can you still create any matrix with this smaller set of elementary matrices? Either show that you still can, or create an $n \times n$ matrix that can no longer be created as a product of this "smaller set" of elementary matrices. In either case, make sure to prove your claim.

4 Bonus Problem

There is a bonus problem here! Again, only attempt this after you've completed absolutely everything else, and are completely sure and confident in your other answers. These are the hardest five points to earn on the test.

1. Can you find four points in $\mathbb{R}^2$, such that the pairwise distance between each of them is odd? Either find such a set of points, or prove this is impossible.