Math 108a | Professor: Padraic Bartlett

Lecture 9: Range and Null Space

Week 3 | *UCSB 2013*

Today's lecture is on the concepts of **range** and **null space**, a pair of concepts related to the linear maps we've been studying recently. We define these two objects below:

1 Range and Null Space.

Definition. Pick two vector spaces U, V. Let $T : U \to V$ be a linear map from U to V.
The **range** of T is the following set:

$$\text{range}(T) = \{T(\vec{u}) \mid \vec{u} \in U\}$$

In other words, the range of a linear map is the collection of all possible outputs of T under all possible inputs from U. Some people call this the **image** of T, and denote this $\text{im}(T)$. Others will denote this as $T(U)$, the idea being that you've put "all" of U into T itself.

Definition. Pick two vector spaces U, V. Let $T : U \to V$ be a linear map from U to V.
The **null space** of T is the following set:

$$\text{null}(T) = \{\vec{u} \mid T(\vec{u}) = \vec{0} \in V\}$$

In other words, the null space of a linear map is the collection of all of the elements in U that T maps to 0.

For example, consider the linear map $T : \mathbb{R}^4 \to \mathbb{R}^2$,

$$T(w, x, y, z) = (0, 0).$$

For this map,

- The **image** of T is the set $\{(0, 0)\}$, because T outputs $(0, 0)$ on every input.
- The **null space** of T is all of $\mathbb{R}^4$, because T sends every element of $\mathbb{R}^4$ to $(0, 0)$.

Similarly, consider the map $T : \mathbb{R}^4 \to \mathbb{R}$, defined such that

$$T(x, y, z) = x + y + z.$$

Thing you should do if you don't believe it: show this is a linear map. Once you've done this, then you can easily check the following:

- The **image** of T is all of $\mathbb{R}$. This is because on input $(a, 0, 0)$, for any real number a, T outputs $a + 0 + 0 = a$. Therefore, we can get any real number as an output of T. Because T's output is restricted to $\mathbb{R}$, there's nothing else to worry about getting; consequently, the image of T is precisely T.

- The **null space** of T is the collection of all triples (a, b, c) such that $T(a, b, c) = a + b + c = 0$. In other words, if we solve for c in terms of the other two variables, it's the collection $\{(a, b, -a-b) : a, b \in \mathbb{R}\}$ of vectors in $\mathbb{R}^3$.

Fun fact: both of these objects are subspaces! We prove this here:

Theorem 1. *The **range** of a linear map $T : U \to V$ is a subspace of V.*

Proof. Let $T : U \to V$ be a linear map, and range(T) denote the range of T: i.e.

$$\text{range}(T) = \{T(\vec{u}) \mid \vec{u} \in U\}$$

We want to show this is a subspace of the set it's contained in, V.

To do this, we recall from week 1 that we simply need to check the following three conditions:

- **Closure**$(+)$: we need to take any two vectors $\vec{x}, \vec{y} \in \text{range}(T)$, and show that their sum is in the range of T.

  Doing this is not very hard. If $\vec{x}, \vec{y}$ are both in the range of T, then by definition there must be a pair of vectors $\vec{a}, \vec{b}$ such that $T(\vec{a}) = \vec{x}, T(\vec{b}) = \vec{y}$. Therefore, the vector $T(\vec{a} + \vec{b})$ is also in the range of T, because it is something else that is an output of T. But $T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b}) = \vec{x} + \vec{y}$. Therefore, given any two vectors $\vec{x}, \vec{y}$ in the range of T, their sum is also in the range of T.

- **Closure**$(\cdot)$: we need to take any vector $\vec{x} \in \text{range}(T)$, and any scalar $\lambda \in F$, and show that their product is in the range of T.

  This is pretty easy, and similar to what we just did above. Again, because $\vec{x}$ is in the range of T, there has to be some vector $\vec{a}$ such that $T(\vec{a}) = \vec{x}$. Then, we have that $T(\lambda\vec{a})$ must also be in the range of T, because it's another possible output of T. But $T(\lambda\vec{a}) = \lambda T(\vec{a}) = \lambda\vec{x}$. Therefore, the range is closed under multiplication by scalars.

- **Identity**$(+)$: Take any $\vec{a} \in U$. Because U is a vector space, we know that $-\vec{u}$ is contained within U. We know that $T(\vec{a} - \vec{a}) = T(\vec{a}) - T(\vec{a}) = \vec{0}$; therefore, $\vec{0}$ is in the range of T.

$\square$

Theorem 2. *The **null space** of a linear map $T : U \to V$ is a subspace of V.*

Proof. Let $T : U \to V$ be a linear map, and null(T) denote the null space of T: i.e.

$$\text{null}(T) = \{\vec{u} \mid T(\vec{u}) = \vec{0} \in V\}$$

We want to show this is a subspace of the set it's contained in, U.

To do this, we recall from week 1 that we simply need to check the following three conditions:

- **Closure**$(+)$: we need to take any two vectors $\vec{x}, \vec{y} \in \text{null}(T)$, and show that their sum is in the null space of T.

  Doing this is easier than for the range! If $\vec{x}, \vec{y}$ are both in the null space of T, then by definition $T(\vec{x}) = T(\vec{y}) = \vec{0}$. But then $T(\vec{x}+\vec{y}) = T(\vec{x})+T(\vec{y}) = \vec{0}+\vec{0} = \vec{0}$. Therefore, given any two vectors $\vec{x}, \vec{y}$ in the null space of T, their sum is also in the null space of T.

- **Closure**$(\cdot)$: we need to take any vector $\vec{x} \in \text{null}(T)$, and any scalar $\lambda \in F$, and show that their product is in the null space of T.

  This is also easy. Again, because $\vec{x}$ is in the null space of T, $T(\vec{x}) = \vec{0}$. Then, we have that $T(\lambda\vec{x}) = \lambda T(\vec{x}) = \lambda\vec{0} = \vec{0}$. Therefore, the null space is closed under multiplication by scalars.

- **Identity**$(+)$: This is identical to the proof for range. Take any $\vec{a} \in U$. Because U is a vector space, we know that $-\vec{u}$ is contained within U. We know that $T(\vec{a} - \vec{a}) = T(\vec{a}) - T(\vec{a}) = \vec{0}$; therefore, $\vec{a} - \vec{a} = \vec{0}$ is in the null space of T.

$\square$

What I want to talk about here, briefly, is why we care about these objects. The motivation for linear maps is perhaps not too hard to understand: whenever we have any collection of objects in mathematics, one of the most natural things to study is functions between these objects. If we want these maps to preserve the structure that we care about with vector spaces — i.e. if we want our map to preserve the operations of vector addition and scalar multiplication, because those are the only two things we can do with vectors — we want this map to play nicely with addition and scalar multiplication, in pretty much precisely the way we ask linear maps to do.

The reason for studying the **range** is not much harder to understand. If we care about what some linear map $T : U \to V$ does, then we pretty clearly care about understanding the range of possible outputs that T has; this is one of the more fundamental things we can ask about T.

The **null space** is more obscure, but arguably the more important of the two. On one hand, understanding the collection of all things that goes to 0 seems somewhat silly; why do we care so much about 0? Why not any other value?

The reason for this comes from understanding how linear maps play with the null space. For example, consider the map

$$T : \mathbb{R}^3 \to \mathbb{R},$$

defined by

$$T(x, y, z) = x + y.$$

The null space of this map is just the collection of all triples (x, y, z) such that

$$T(x, y, z) = 0;$$

i.e. it's the set

$$\text{null}(T) = \{(x, -x, z) : x, z \in \mathbb{R}\}.$$

So, here's a related question. What does the set of all vectors that map to 1 look like? Well, if we directly solve, we're looking for all triples (x, y, z) such that

$$T(x, y, z) = 1;$$

i.e. it's the set

$$\{(1 + x, -x, z) : x, z \in \mathbb{R}\}.$$

In other words, it's basically what happens if we take null(T) and scale every element in it by $(1, 0, 0)$!

Furthermore, if we take **any** real number $a \in \mathbb{R}$. we can see that

$$T(x, y, z) = a$$

if and only if our triple has the form

$$(a + x, -x, z),$$

for some $x, z \in \mathbb{R}$.

So, in a sense, when we understood the null space of the linear map T above, for any a we understood the collection of **all** elements that map to that a! So there's nothing special about 0, in a sense — rather, the null space appears to be capturing the total "redundancy" of our map, i.e. the number of elements that our maps sends to **any** element!

We'll study this phenomena more in later lectures.