Math 108a | Professor: Padraic Bartlett

Lecture 8: Linear Maps

Week 3 | *UCSB 2013*

In our first week, we studied fields and vector spaces as abstract objects. In our second week, we transitioned from this abstract view to a more concrete and hands-on approach: instead of proving that certain properties were held by all vector spaces, we started looking at things like linear independence, basis and dimension that we could study and examine for specific individual vector spaces.

At this point, while there are still more vector space properties we could explore, we turn our attention instead to what we can **do** with these vector spaces. In other words: we've studied vector spaces for over two weeks now. We understand them decently well. What can we do with this knowledge?

In today's talk, we start exploring this question, with the specific target of understanding **linear maps.** We define linear maps in the following section:

1 Linear Maps: Definitions and Examples

Definition. A **linear map** from a vector space V to another vector space W, where V and W are two vector spaces over some field F, is a function $T : V \to W$ with the following properties:

- **Plays well with addition**: for any $\vec{v}, \vec{w} \in V$, $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$.
- **Plays well with multiplication**: for any $\vec{v} \in V$ and any $a \in \mathbb{R}$, $T(a\vec{v}) = aT(\vec{v})$.

If people are being formal, they will say that maps that satisfy the first property are "additive," while maps that satisfy the second property are"homogeneous."

Also, sometimes people call linear maps linear transformations. These are the same things.

For example, the map $id : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$id(x, y) = (x, y)$$

is a linear map, because

- **it plays well with addition**: for any two vectors $(a, b),(c, d)$, we have $id((a, b) + (c, d)) = id(a + c, b + d) = (a + c, b + d)$. This is the same thing as $id(a, b) + id(c, d) = (a, b) + (c, d) = (a + c, b + d)$.
- **it plays well with multiplication**: for any vector (a, b) and any real number λ, we have $id(\lambda(a, b)) = id(\lambda a, \lambda b) = (\lambda a, \lambda b)$. This is the same thing as $\lambda\ id(a, b) = \lambda(a, b) = (\lambda a, \lambda b)$.

Conversely, the map $T : \mathcal{P}_1(\mathbb{R}) \to \mathcal{P}_1(\mathbb{R})$, defined by

$$T(a + bx) = a^2$$

is not a linear map, because

- **it does not play well with addition**. Specifically, look at the two polynomials $2, 2 + x$ in $\mathcal{P}_1(\mathbb{R})$. $T(2 + (2 + x)) = T(4 + x) = 4^2 = 16$, while $T(2) + T(2 + x) = 2^2 + 2^2 = 8$.

Essentially, we care about linear maps because they're a way to take one vector space and "turn it" somehow into another vector space: i.e. it gives us a way of taking something that has addition and scalar multiplication operations and turning it into something else that has addition and scalar multiplication operations, in a way that "preserves" these two operations (because it "plays well" with addition and scalar multiplication.) We'll get more into these ideas in our next class. For now, however, we simply examine a large family of maps, and for each one decide if it is linear or not.

1. $T : \mathbb{R} \to \mathbb{R}$, defined such that

   $$T(x) = |x|.$$

   Claim. This is not a linear map.

   Proof. **This map does not play well with addition**. Specifically, notice that $T(1 + (-1)) = T(0) = 0$, while $T(1) + T(-1) = |1| + |-1| = 2$. These two quantities are different; therefore, our map is not additive. □

2. $T : \mathbb{R}^4 \to \mathbb{R}^2$, defined such that

   $$T(w, x, y, z) = (0, 0).$$

   Claim. This is a linear map.

   Proof. We simply show that it satisfies the following two properties:

   - **It plays well with addition**. Take any two vectors $\vec{v}, \vec{w} \in \mathbb{R}^4$. We know that $T(\vec{v} + \vec{w}) = (0, 0)$, by definition. Similarly, we know that $T(\vec{v}) + T(\vec{w}) = (0, 0) + (0, 0) = (0, 0)$. Therefore, these two quantities are the same for any two vectors $\vec{v}, \vec{w}$; therefore, our map is additive.
   - **It plays well with multiplication**. Take any vector $\vec{v} \in \mathbb{R}^4$ and any $\lambda \in \mathbb{R}$. On one hand, we know that $T(\lambda\vec{v}) = (0, 0)$ by definition; on the other, we know that $\lambda T(\vec{v}) = \lambda(0, 0) = (0, 0)$. Therefore, this map is homogenous.

   □

3. $T : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}$, defined such that

   $$T(p(x)) = p(1).$$

Claim. This is a linear map.

Proof. We simply show that it satisfies the following two properties:

- **It plays well with addition**. Take any two polynomials $p(x), q(x) \in \mathcal{P}_3(\mathbb{R})$. We know that $T(p(x)+q(x))$ is just the polynomial $p(x)+q(x)$ evaluated at $x = 1$, which is just $p(1) + q(1)$. On the other hand, $T(p(x)) + T(q(x)) = p(1) + q(1)$. These are equal; therefore, our map is additive.
- **It plays well with multiplication**. Take any polynomial $p(x) \in \mathcal{P}_3(\mathbb{R})$, and $\lambda \in \mathbb{R}$. We know that $T(\lambda p(x)) = \lambda \cdot p(1)$, and that $\lambda T(p(x)) = \lambda p(1)$. These two quantities are equal; therefore, this map is homogenous.

□

4. $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_5(\mathbb{R})$, defined such that

$$T(p(x)) = (1 + x) \cdot p(x).$$

Claim. This is a linear map.

Proof. We simply show that it satisfies the following two properties:

- **It plays well with addition**. Take any two polynomials $p(x), q(x) \in \mathcal{P}_2(\mathbb{R})$. We know that $T(p(x)+q(x))$ is just the polynomial $(1+x)\cdot(p(x)+q(x))$, which can be expanded as $(1+x)p(x)+(1+x)q(x)$. On the other hand, $T(p(x))+T(q(x)) = (1 + x)p(x) + (1 + x)q(x)$. These are equal; therefore, our map is additive.
- **It plays well with multiplication**. Take any polynomial $p(x) \in \mathcal{P}_2(\mathbb{R})$, and $\lambda \in \mathbb{R}$. We know that $T(\lambda p(x)) = (1 + x)\lambda p(x) = \lambda(1 + x)p(x)$, and that $\lambda T(p(x)) = \lambda(1 + x)p(x)$. Because these two quantities are equal, this map is homogenous.

□

5. $T : \mathbb{R}^n \to \mathbb{R}^{n-1}$, defined such that

$$T(x_1, \ldots x_n) = (x_2, x_3, \ldots x_n).$$

Claim. This is a linear map.

Proof. We simply show that it satisfies the following two properties:

- **It plays well with addition**. Take any two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$. We know that $T(\vec{v} + \vec{w}) = T(v_1 + w_1, v_2 + w_2, \ldots v_n + w_n) = (v_2 + w_2, v_3 + w_3, \ldots v_n + w_n)$. Similarly, we know that $T(\vec{v}) + T(\vec{w}) = (v_2, v_3, \ldots v_n) + (w_2, w_3, \ldots w_n) = (v_2 + w_2, v_3 + w_3, \ldots v_n + w_n)$. Therefore, these two quantities are the same for any two vectors $\vec{v}, \vec{w}$; therefore, our map is additive.

- **It plays well with multiplication**. Take any vector $\vec{v} \in \mathbb{R}^n$ and any $\lambda \in \mathbb{R}$. On one hand, we know that $T(\lambda\vec{v}) = T(\lambda v_1, \lambda v_2, \ldots \lambda v_n) = (\lambda v_2, \lambda v_3, \ldots \lambda v_n)$ by definition; on the other, we know that $\lambda T(\vec{v}) = \lambda(v_2, v_3, \ldots v_n) = (\lambda v_2, \lambda v_3, \ldots \lambda v_n)$. Therefore, this map is homogenous.

□

6. $T : \mathbb{R}^3 \to \mathcal{P}_3(\mathbb{R})$, defined such that

$$T(a, b, c) = (x - a) \cdot (x - b) \cdot (x - c).$$

Claim. This is not a linear map.

Proof. **This map does not play well with multiplication**. Specifically, notice that $T(2(1,1,1)) = T(2,2,2) = (x-2)(x-2)(x-2) = (x-2)^3$, while $2(T(1,1,1)) = 2(x-1)(x-1)(x-1) = 2(x-1)^3$. These two quantities are different; in particular, the first polynomial has all of its roots at 2, while the second has all of its roots at 1. Therefore, our map is not homogenous. □

7. $T : \mathcal{P}_4(\mathbb{R}) \to T : \mathcal{P}_3(\mathbb{R})$, defined such that

$$T(p(x)) = \frac{d}{dx} p(x).$$

Claim. This is a linear map.

Proof. We simply show that it satisfies the following two properties:

- **It plays well with addition**. Take any two polynomials $p(x), q(x) \in \mathcal{P}_4(\mathbb{R})$. We know that $T(p(x) + q(x))$ is just $(\frac{d}{dx}(p(x) + q(x))$, which can be expanded as $\frac{d}{dx}p(x) + \frac{d}{dx}q(x)$. On the other hand, $T(p(x)) + T(q(x)) = \frac{d}{dx}p(x) + \frac{d}{dx}q(x)$. These are equal; therefore, our map is additive.
- **It plays well with multiplication**. Take any polynomial $p(x) \in \mathcal{P}_4(\mathbb{R})$, and $\lambda \in \mathbb{R}$. We know that $T(\lambda p(x)) = \frac{d}{dx}\lambda p(x) = \lambda \frac{d}{dx} p(x)$, and that $\lambda T(p(x)) = \lambda \frac{d}{dx} p(x)$. Because these two quantities are equal, this map is homogenous.

□

8. $T : \mathbb{R}^4 \to \mathbb{R}^2$, defined such that

$$T(w, x, y, z) = (6w + 73x, 42y + 23z)\,.$$

Proof. We simply show that it satisfies the following two properties:

- **It plays well with addition**. Take any two vectors $(w, x, y, z), (a, b, c, d)$ in $\mathbb{R}^4$. Then $T((a, b, c, d) + (w, x, y, z)) = T(a+w, b+x, c+y, d+z) = (6(a+w)+73(b+x), 42(c+y)+23(d+z))$. On the other hand, $T(a, b, c, d) + T(w, x, y, z) = (6a+73b, 42c+23d) + (6w+73x, 42y+23z) = (6(a+w)+73(b+x), 42(c+y)+23(d+z))$. These are equal; therefore, our map is additive.

- **It plays well with multiplication**. Take any vector $(w, x, y, z) \in \mathbb{R}^4$ and any $\lambda \in \mathbb{R}$. On one hand, we know that $T(\lambda(w, x, y, z)) = T(\lambda w, \lambda x, \lambda y, \lambda z) = (6\lambda w + 73\lambda x, 42\lambda y + 23\lambda z)$. On the other, we know that $\lambda T(w, x, y, z) = \lambda(6w + 73x, 42y + 23z) = (6\lambda w + 73\lambda x, 42\lambda y + 23\lambda z)$. Therefore, this map is homogenous.

□

9. $T : \mathbb{R}^4 \to \mathbb{R}^2$, defined such that

$$T(w, x, y, z) = (wx, yz).$$

Claim. This is not a linear map.

Proof. **This map does not play well with multiplication**. Specifically, notice that $T(2(1, 1, 1, 1)) = T(2, 2, 2, 2) = (4, 4)$, while $2(T(1, 1, 1, 1)) = 2(1, 1) = (2, 2)$. These two quantities are different; therefore, our map is not homogenous. □

10. $T : \mathbb{R}^2 \to \mathbb{R}$, defined such that

$$T(x, y) = \left(x^3 + y^3\right)^{1/3}.$$

Claim. This is not a linear map.

Proof. This map is somewhat odder than the others, as it **does** play well with multiplication. However, **this map does not play well with addition**. Specifically, notice that

$$T(1, 0) + T(0, 1) = \left(1^3 + 0^3\right)^{1/3} + \left(0^3 + 1^3\right)^{1/3} = 1 + 1 = 2,$$

while

$$T((1, 0) + T(0, 1)) = T(1, 1) = \left(1^3 + 1^3\right)^{1/3} = 2^{1/3}.$$

□

Fun bonus question: can you find a map that is additive (i.e. plays well with addition), but not homogenous (i.e. does not play well with multiplication)?