Math 108a	Professor: Padraic Bartlett

Lecture 6: Basis and Dimension

Week 2	*UCSB 2013*

In our last talk, we introduced the concepts of **span** and **linear independence**. We continue introducing new vector space concepts with today's pair of definitions: the concepts of **basis** and **dimension**.

1 Basis

We closed our talk Monday by proving the following theorem:

Theorem 1. *Any finite set of vectors S has a linearly independent subset T, such that span(S) = span(T).*

The motivation for this theorem was the desire to take a set S and "remove" all of the elements that aren't necessary when we construct span(S). I.e. if a set S was linearly dependent, we showed that this meant that one of its vectors $\vec{v}$ can be written as a linear combination of other elements of S. Therefore, in a sense, this vector $\vec{v}$ is "superfluous" with respect to the span of S: we could remove it without changing anything!

This idea — of a set S that doesn't have any redundancy in it, like the ones created by our theorem 1 — is a valuable one in linear algebra. Accordingly, we have a term for these kinds of sets:

Definition. Take a vector space V. A **basis** B for V is a set of vectors B such that B is linearly independent, and span$(B) = V$.

Bases are really useful things. You're already aware of a few bases:

- The set of vectors $\vec{e_1} = (1, 0, 0 \ldots 0), e_2 = (0, 1, 0 \ldots 0), \ldots e_n = (0, 0 \ldots 0, 1)$ is a basis for $\mathbb{R}^n$.
- The set of polynomials $1, x, x^2, x^3, \ldots$ is a basis for $\mathbb{R}[x]$.

As a quick example, we study another interesting basis:

Question. Consider the set of vectors

$$S = \{(1, 1, 1, 1), (1, 1, -1, -1), (1, -1, 1, -1), (1, -1, -1, 1)\}.$$

Show that this is a basis for $\mathbb{R}^n$.

Proof. Take any $(w, x, y, z) \in \mathbb{R}^4$. We want to show that there are always a, b, c, d such that

$$a(1, 1, 1, 1) + b(1, 1, -1, -1) + c(1, -1, 1, -1) + d(1, -1, -1, 1) = (w, x, y, z),$$

and furthermore that if $(w, x, y, z) = (0, 0, 0, 0)$ that this forces a, b, c, d to all be 0. This proves that the span of S is all of $\mathbb{R}^4$ and that S is linearly independent, respectively.

We turn the equation above into four equalities, one for each coördinate in $\mathbb{R}^4$:

$$\begin{aligned} a+b+c+d &= w \\ a+b-c-d &= x \\ a-b+c-d &= y \\ a-b-c+d &= z \end{aligned}$$

Summing all four equations gives us

$$4a = w+x+y+z.$$

Adding the first two equations and subtracing the second two equations gives us

$$4b = w+x-y-z.$$

Adding the first and third, and subtracting the second and fourth gives us

$$4c = w+y-x-z.$$

Finally, adding the first and fourth and subtracting the second and third yields

$$4d = w+z-x-y.$$

So: if $(w,x,y,z) = (0,0,0,0)$, this means that $a=b=c=d=0$. Therefore, our set is linearly independent.

Furthermore, for any (w,x,y,z), we have that

$$\begin{aligned} &\frac{w+x+y+z}{4}(1,1,1,1) + \frac{w+x-y-z}{4}(1,1,-1,-1) \\ +&\frac{w+y-x-z}{4}(1,-1,1,-1) + \frac{w+z-x-y}{4}(1,-1,-1,1) = (w,x,y,z). \end{aligned}$$

Therefore, we can combine these four elements to get any vector in $\mathbb{R}^4$; i.e. our set spans $\mathbb{R}^4$. $\square$

This example is interesting because its entries satisfy the following two properties:

- Every vector is made up out of entries from ± 1.
- The dot product of any two vectors is 0.

Finding a basis of vectors that can do this is actually an open question. We know that they exist for any $\mathbb{R}^n$ where n is a multiple of 4 up to 664, but no one's found such a basis for $\mathbb{R}^{668}$. Find one for extra credit?

Another natural idea to wonder about is the following: given a vector space V, what is the smallest number of elements we need to make a basis? Can we have two bases with different lengths?

This is answered in the following theorem:

Theorem. Suppose that V is a vector space with two bases $B_1 = \{\vec{v_1}, \dots \vec{v_n}\}$, $B_2 = \{\vec{w_1}, \dots \vec{w_m}\}$ both containing finitely many elements.. Then these sets have the same size: i.e. $|B_1| = |B_2|$.

Proof. Take any two sets $B_1 = \{\vec{v_1}, \dots \vec{v_n}\}$, $B_2 = \{\vec{w_1}, \dots \vec{w_m}\}$ such that

- B_1, B_2 span V.
- B_1, B_2 are linearly independent.

We will show that these two sets must be the same size.

To do this, pick any vector $\vec{v_1} \in B_1$. Use the fact that B_2 spans V to write $\vec{v_1}$ as a linear combination of elements in B_2. I.e. find constants a_i such that

$$\vec{v_1} = a_1\vec{w_1} + \dots a_n\vec{w_n}$$

Because $\vec{v_1}$ is nonzero, there is some a_j such that $a_j \neq 0$. Consequently, we can take this equality and solve for $\vec{w_j}$:

$$\vec{w_j} = \frac{-1}{a_j}\left(a_1\vec{w_1} + \dots + a_{j-1}\vec{w_{j-1}} + a_{j+1}\vec{w_{j+1}} + \dots + a_n\vec{w_n} + \vec{v_1}\right).$$

Therefore, we have that $\vec{w_j}$ is in the span of the set $B_2' = \{\vec{w_1}, \dots \vec{w_{j-1}}, \vec{w_{j+1}}, \dots \vec{w_n}, \vec{v_1}\}$. We also have all of the other $\vec{w_i}$'s in this set: therefore, we have all of B_2 in this span! In other words, the span of this B_2' is all of V, just like the span of B_2! Essentially, we've shown that we can "replace" one of the $\vec{w_j}$ vectors with one of the elements from B_1.

Moreover, we know that this set is still linearly independent. To see this, notice the following things:

- There is only one way to write $\vec{v_1}$ as a sum of elements in B_2. (If there were two different ways, then their difference would be a nontrivial combination of elements in B_2 that sums to 0. B_2 is linearly independent, so that's impossible.
- If we have any linear combination of elements in B_2' that sums to 0, if it does not use $\vec{v_1}$, then it must be trivial (i.e. all scalars are 0) because B_2 is linearly independent.
- So if we have a linear combination of elements in B_2' that sums to 0, it must use $\vec{v_1}$ nontrivially. Using this, we can simply solve for $\vec{v_1}$ in terms of the other vectors. This combination does not use $\vec{w_j}$, because that vector is not in B_2': which means we have found a second way to write $\vec{v_1}$ as a sum of elements in B_2. But we said that was impossible!

We repeat this trick with $\vec{v_2}$: i.e. we find a combination of vectors in B_2' that yields $\vec{v_2}$. This combination cannot consist only of $\vec{v_1}$, because we know that B_1 is linearly independent, and therefore that there is no nontrivial way to combine some of the elements of B_1 to get another element of B_1. Therefore, there must be some $a_i\vec{w_i}$ used in this linear combination with a_i nonzero; again, solve for $\vec{w_i}$ and use this observation to "replace" $\vec{w_i}$ with $\vec{v_2}$. Call this new set $B_2^{(2)}$. Again, this set is linearly independent, for the same reasons as before.

Keep doing this. As stated before, we can always find a way to express each $\vec{v_{k+1}}$ as a linear combination of elements in $B_2^{(k)}$ because these sets span our whole vector space; moreover, these combinations always involve an element $a_i\vec{w_i}$ because the set B_1 is linearly independent.

Therefore, the only way this process stops is when we run out of elements in B_1. When this happens, look at the set $B_2^{(n)}$ that we get. We've placed all of the elements of B_1 in this set. If there was an element $\vec{w_k}$ of B_2 left in this set, then our set would be linearly dependent: this is because B_1 spans all of V, and therefore we can express $\vec{w_k}$ as some combination of elements in B_2.

But this is impossible: we showed that these sets $S^{(k)}$ are always linearly independent! Therefore, there are no elements of B_2 left in this set. Because we got rid of elements one at a time and stopped after n steps, this means that there are n elements in B_2. In other words, B_2 and B_1 have the same number of elements. $\square$

Using this, we can finally define the concept of **dimension**:

Definition. Suppose that V is a vector space with a basis B containing finitely many elements. Then we say that the **dimension** of V is the number of elements in B.

For example, the dimension of $\mathbb{R}^n$ is n, because this vector space is spanned by the vectors $\vec{e_1} = (1, 0, 0 \ldots 0), e_2 = (0, 1, 0 \ldots 0), \ldots e_n = (0, 0 \ldots 0, 1)$.