Math 108a | Professor: Padraic Bartlett

Lecture 5: Span and Linear Independence

Week 2 | *UCSB 2013*

Our lectures thus far have focused on the two concepts of **fields** and **vector spaces**. In specific, we've studied these two objects in a fairly formal setting: we've created sets of axioms that describe these objects, used these axioms to deduce certain properties that all fields and vector spaces must possess, and built up a collection of useful examples of both fields ($\mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}/2\mathbb{Z}$) and vector spaces ($\mathbb{R}^n, \mathbb{C}^n, \mathbb{R}[x], M_{\mathbb{R}}(n,n)$, and a bunch of subspaces of these spaces.)

In this next sequence of lectures, we're going to change our focus somewhat. Now that we've built up a good set of examples, the natural thing to ask is what properties set them apart? What makes them interesting? What sorts of properties can we study for each individual space?

We study two such objects in this talk: the concepts of **span** and **linear independence**.

1 Span and Linear Independence

When we're working with vectors in a vector space, there's really only two operations we've discussed that we can do: vector addition and scalar multiplication. A useful thing to think about, then, is the collection of all sorts of objects that we can make using these operations! We define these things here:

Definition. Let V be a vector space over some field F. A **linear combination** of some set of vectors S is any sum of the form

$$a_1\vec{v_1} + a_2\vec{v_2} + \ldots a_n\vec{v_n},$$

where $a_1, \ldots a_n$ are all elements of our field F, and $\vec{v_1}, \ldots \vec{v_n} \in S$.

In other words, a linear combination of some set of vectors is anything we can make by scaling and adding these vectors together.

A useful thing to study, given some set $\{\vec{v_1}, \ldots \vec{v_n}$ of vectors, is the following: what can we make with these vectors? In other words, what is the collection of all **linear combinations** of these vectors?

This question is sufficiently common that we have a term for its related concept: the idea of **span**!

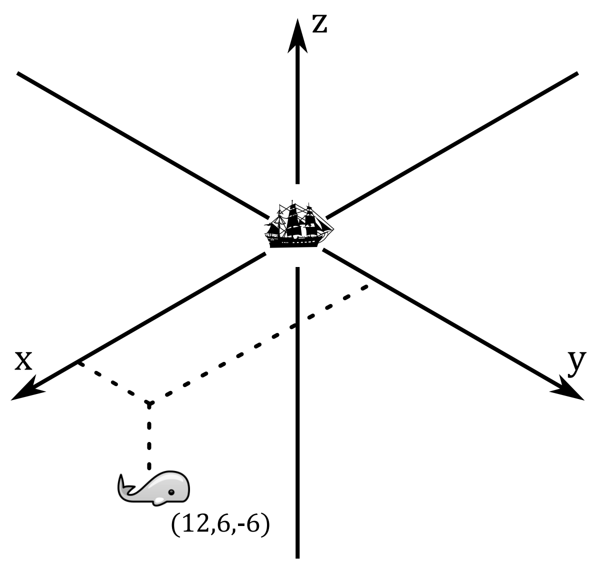
Definition. Given any collection of vectors A, the **span** of A, denoted span(A), is the collection of all linear combinations of elements of A. In other words,

$$\text{span}(A) = \{a_1\vec{v_1} + a_2\vec{v_2} + \ldots a_n\vec{v_n} | n \in \mathbb{N}, a_1 \ldots a_n \in \mathbb{R}, \vec{v_1}, \ldots \vec{v_n} \in A\}.$$

A genre of questions you'll often encounter in a linear algebra class is "Given some set A of vectors, what is their span? Is some other vector $\vec{w}$ contained within this span?"

We give an example of a calculation here:

Question. You have been hired as a deep-sea navigator. Your first task is to pilot a submarine from our ship, anchored at $(0, 0, 0)$ in the diagram below, to observe a whale at $(12, 6, -6)$. Your submarine has three engines on it. The first, when ran for a cycle, moves the submarine $(2, 1, 0)$ units from its current location. The second, when ran for a cycle, moves the submarine $(0, 2, 1)$ units from its current location. Finally, the third, when ran for a cycle, moves the submarine $(1, 0, 2)$ units from its current location. Submarine engines can be fired for fractional units of time, and when ran in reverse moves the shuttle backwards along that given vector.



Can you make it to the whale?

Answer. Essentially, our question is asking if the vector $(12, 6, -6)$ is contained in the span of the three vectors $(2, 1, 0), (0, 2, 1), (1, 0, 2)$.

I claim that it is! To see why, simply just start trying to combine these three vectors into $(12, 6, -6)$. If we assume that we fire the first engine for a units, the second for b units, and the third for c units, we're essentially trying to find a, b, c such that

$$a(2, 1, 0) + b(0, 2, 1) + c(1, 0, 2) = (12, 6, -6).$$

This gives us three equations, one for the x-coördinate, one for the y-coördinate, and a third for the z-coördinate:

$$\begin{aligned} 2a + 0b + 1c &= 12, \\ 1a + 2b + 0c &= 6, \\ 0a + 1b + 2c &= -6. \end{aligned}$$

Subtracting two copies of the second equation from the first gives us

$$-4b + c = 0,$$

in other words $c = 4b$. Plugging this into the last equation gives us

$$b + 2(4b) = -6,$$

i.e. $b = -\frac{2}{3}$. This gives us then that $c = -\frac{8}{3}$, and thus that

$$2a - \frac{8}{3} = 12,$$

i.e. $a = \frac{22}{3}$.

Consequently, we've just calculated that

$$\frac{22}{3}(2,1,0) - \frac{2}{3}(0,2,1) - \frac{8}{3}(1,0,2) = (12,6,-6);$$

in other words, that $(12,6,-6)$ is in the span of our set of vectors, and therefore that we can get to it by using our three engines as described by a, b, c!

A fact worth noting is the following:

Proposition. Let V be a vector space over a field F and A any nonempty subset of vectors of V. The span of A, $\text{span}(A)$, is a subspace of V.

Proof. As on Friday, we just have to check three properties:

- **Closure(+)**: Take any two vectors $\vec{x}, \vec{y} \in \text{span}(A)$. For each of these vectors, because they're in the span of A, we can write them as some linear combination of elements of A: i.e. we can find field elements $a_1, \ldots a_n, b_1, \ldots b_m \in F$ and vectors $\vec{v_1}, \ldots \vec{v_n}, \vec{w_1}, \ldots \vec{w_m} \in A$ such that

$$a_1\vec{v_1} + \ldots a_n\vec{v_n} = \vec{x},$$
$$b_1\vec{w_1} + \ldots b_n\vec{w_n} = \vec{y}.$$

  Then, we have that

$$a_1\vec{v_1} + \ldots a_n\vec{v_n} + b_1\vec{w_1} + \ldots b_n\vec{w_n} = \vec{x} + \vec{y}.$$

  In other words, $\vec{x} + \vec{y}$ is a linear combination of elements in A! Therefore, $\vec{x} + \vec{y}$ is also in the span of A.

- **Closure(·)**: Take any vector $\vec{x} \in \text{span}(A)$. Like before, because $\vec{x}$ is in the span of A, we can write it as some linear combination of elements of A: i.e. we can find field elements $a_1, \ldots a_n \in F$ and vectors $\vec{v_1}, \ldots \vec{v_n} \in A$ such that

$$a_1\vec{v_1} + \ldots a_n\vec{v_n} = \vec{x}.$$

  Pick any $b \in F$. We have that

$$b\vec{x} = b\left(a_1\vec{v_1} + \ldots a_n\vec{v_n}\right) = ba_1\vec{v_1} + \ldots ba_n\vec{v_n};$$

  in other words, $b\vec{x}$ is a linear combination of elements of A, and is therefore in the span of A.

- **Identity(+)**. Take any vector $\vec{v} \in A$. We know/have proven that $0\vec{v} = \vec{0}$, the additive identity, for any vector space V. Therefore, because span(A) is closed under scalar multiplication, we have just shown that $\vec{0}$ is in span(A).

Because these three properties hold, by our results on Friday (which said that given these three properties along with the fact that span(A) is a subset of V, the rest of the vector space properties must follow) we have just shown that this is a subspace of V! □

Based off of this observation, we can make the following definition:

Definition. Suppose that $A = \{\vec{v}_1, \ldots \vec{v}_n\}$ is some set of vectors, drawn from some vector space V. In the proposition above, we proved that span(A) is a subspace of the vector space V: in other words, span(A) is a vector space in its own right! Suppose that span(A) is equal to the vector space U. Then we say that A **spans** U.

This definition motivates a second kind of question: take some vector space V. Can we find a set A of vectors that spans V?

The answer here is clearly yes: we can just pick A to be V itself! Then the span of A is certainly V, because every vector of V is simply contained in A. Yet, this answer is also kind of dumb. While A spans V, it does so with a lot of redundancy: i.e. if V was $\mathbb{R}^3$, we'd be using a set with infinitely many elements to span $\mathbb{R}^3$, when we really only need the three vectors $(1,0,0), (0,1,0), (0,0,1)$.

Here's a related question, to help us spot this kind of inefficiency: suppose we have some set A of vectors. Is it possible to remove a vector from A and still have a set with the same span as A?

This concept **also** comes up all the time in mathematics! Therefore, we have a definition for it:

Definition. Let V be some vector space over a field F, and A be a subset of V. We say that the set A is **linearly dependent** if there is some $n > 0$ and distinct elements $\vec{v_1}, \ldots \vec{v_n} \in A$, field coefficients $a_1, \ldots a_n \neq 0 \in F$ such that

$$\vec{0} = a_1\vec{v_1} + a_2\vec{v_2} + \ldots a_n\vec{v_n}.$$

In other words, a set A is linearly dependent if there is a linear combination of elements in A that sums to 0.

If no such combination exists, then we say that A is **linearly independent.**

Notice that if a set is **linearly dependent**, then there is some vector within it that we can remove without changing its span! We prove this here:

Lemma 1. *Let V be a vector space over a field F, and S be a linearly dependent subset of V. Then there is some vector $\vec{v}$ in S that we can remove from S without changing its span.*

Proof. By definition, because S is linearly dependent, there is some $n > 0$ and distinct elements $\vec{v_1}, \ldots \vec{v_n} \in A$, field coefficients $a_1, \ldots a_n \neq 0 \in F$ such that

$$\vec{0} = a_1\vec{v_1} + a_2\vec{v_2} + \ldots a_n\vec{v_n}.$$

Solve for $\vec{v_1}$:

$$\vec{v_1} = -\frac{a_2}{a_1}\vec{v_2} - \frac{a_3}{a_1}\vec{v_2} - \ldots \frac{a_n}{a_1}\vec{v_n}.$$

This is a way to create the vector $\vec{v_1}$ without using the vector $\vec{v_1}$ itself.

Consider the set $S \setminus \{\vec{v_1}\}$, the set created by removing $\vec{v_1}$ from S. We claim that this set has the same span as S itself. To see why, take any element $\vec{u}$ in the span of S: i.e. any linear combination of elements in S. Pick distinct vectors $\vec{w_1}, \ldots \vec{w_m} \in S$ and scalars $b_1, \ldots b_m$ such that

$$\vec{u} = b_1\vec{w_1} + \ldots b_n\vec{w_n}.$$

If none of these vectors $\vec{w_1}, \ldots \vec{w_n}$ are the vector $\vec{v_1}$, then this linear combination is also in the span of $S \setminus \{\vec{v_1}\}$. Otherwise, if some $\vec{w_i} = \vec{v_1}$, then we can just replace $\vec{w_i}$ with

$$-\frac{a_2}{a_1}\vec{v_2} - \frac{a_3}{a_1}\vec{v_2} - \ldots \frac{a_n}{a_1}\vec{v_n}.$$

This gives us a way to combine elements of $S \setminus \{\vec{v_1}\}$ in a way to create $\vec{u}$.

So, we've shown that any element of the span of S is also in the span of $S \setminus \{\vec{v_1}\}$! Therefore, these two sets have the same span, and we've proven our claim: there is a vector, $\vec{v_1}$, that we can remove from S without changing its span. □

By repeatedly applying Lemma 1, we can get the following theorem:

Theorem 2. *Any finite set of vectors S has a linearly independent subset T, such that span(S) = span(T).*

Proof. Take any set S. If it is linearly independent, stop; we're done. Otherwise, repeatedly apply Lemma 1 to find an element we can remove from S without changing its span, and remove that element from S. Because our set is finite, and any set with one vector is linearly independent (prove this if you don't see why!), we know that this process will eventually stop and leave us with a linearly independent set at some stage!

So we've found a subset T of S with the same span as S, that's linearly independent. Yay! □

Side note. In the infinite case, things are much more complicated. In particular, the above process, of repeatedly removing elements one-by-one, isn't guaranteed to ever stop! For example, suppose you had the set $S = \{(z, 0) : z \neq 0 \in \mathbb{Z}\}$.

This is an infinite set. Moreover, because the span of this set is simply $\{(x, 0) : x \in \mathbb{R}\}$, literally any single element $(z, 0)$ of S will have the same span as S: this is because $\frac{x}{z}(z, 0)$ is a linear combination using just $(z, 0)$ that can generate any element in span(S).

However, suppose you were simply removing elements one-by-one from S. It is possible, through sheer bad luck, you would pick the elements $(1, 0), (2, 0), (3, 0), (4, 0), \ldots$ all in a row. This would never get you down to a linearly independent set! In particular, you'd still have all of the elements $(z, 0)$ where z is a negative integer lying around, and there are tons of linear combinations of these elements that combine to $(0, 0)$.

So the argument above doesn't work. However, the result is still true: there **is** a subset that is linearly independent with the same span! Proving that this always exists, though, is tricky, and requires the use of things like the axiom of choice, which is beyond the scope of this class.

To illustrate how we concretely find linearly independent subsets of certain sets of vectors, we work an example:

Question. Consider the set

$$S = \{a, a+1, a+2) : a \in \mathbb{N}, 0 < a \leq 100\}.$$

Is this set linearly dependent? If it is, find a linearly independent subset of this set. If not, prove it is linearly independent.

Proof. So, it certainly seems like this set should be linearly dependent: there are a hundred vectors in the set, and it's a subset of $\mathbb{R}^3$! However, we need to actually prove this is true, which is a little tricker: where do we start?

One good tactic here is to look at some concrete elements of our set, and see what we can do with those elements. (In general, this is a useful tactic when you're presented with a large collection of objects all described using rules: make a few examples and see what those examples do!)

In particular, we know that $(1, 2, 3), (2, 3, 4), (3, 4, 5)$, and $(4, 5, 6)$ are all elements of our set. Is this smaller, easier-to-understand set linearly dependent?

Well: if it were, we would have to have some way of combining elements to get to $(0, 0, 0)$. It's not obvious how to do this as written, though. One strategy here is to just algebra-bash out a solution: i.e. try to find a, b, c, d such that $a(1, 2, 3) + b(2, 3, 4) + c(3, 4, 5) + d(4, 5, 6) = (0, 0, 0)$. This will work! But it seems tedious; solving three equations in four variables is not the most enjoyable thing in the world to do. Also, we've already given an example (earlier, when we were showing that a given vector is in the span of some set) that basically goes through this method.

An alternate strategy, that is sometimes worth pursuing, is not to just try to find combinations of $(1, 2, 3), (2, 3, 4), (3, 4, 5), (4, 5, 6)$ that get to $(0, 0, 0)$, but rather to just look for combinations that get to something **useful**! I.e. if we could make the vectors $(1, 0, 0)$ or $(0, 1, 0)$ or $(0, 0, 1)$, that would be really useful for making $(0, 0, 0)$ later, as these are relatively useful and easy-to-understand vectors!

This, we **can** do. In particular, notice that

$$(2, 3, 4) - (1, 2, 3) = (1, 1, 1),$$

which certainly looks like a useful vector!

In fact, it's great for generating the other vectors in this set! Notice that, for example,

$$(3, 4, 5) = (1, 2, 3) + 2(1, 1, 1) = (1, 2, 3) + 2((2, 3, 4) - (1, 2, 3)),$$

and therefore that we can write $(3, 4, 5)$ as a linear combination of vectors in our set!

More generally, notice that

$$(a, a+1, a+2) = (1, 2, 3) + (a-1)(1, 1, 1) = (1, 2, 3) + (a-1)((2, 3, 4) - (1, 2, 3)),$$

and therefore that we can write **any** vector in our set as a linear combination of the two vectors $(1, 2, 3)$ and $(2, 3, 4)$!

Therefore, we've shown that the span of our set S is just the span of the two vectors $(1, 2, 3)$ and $(2, 3, 4)$. These vectors are clearly linearly independent: to see why, notice that if

$$x(1, 2, 3) + y(2, 3, 4) = (0, 0, 0),$$

we would have to have

$$\begin{aligned} x + 2y &= 0, \\ 2x + 3y &= 0, \text{ and} \\ 3x + 4y &= 0. \end{aligned}$$

The first two equations tell us that $x = -2y$ and that $x = -\frac{3}{2}y$, which can only hold if $x = y = 0$. Therefore, there is no nontrivial combination (i.e. no combination in which the coefficients are nonzero) of these two vectors that is equal to $(0, 0, 0)$.

So we've created a linearly independent subset of S that has the same span as S! Yay, success. $\square$