Math 108a | Professor: Padraic Bartlett

# Lecture 17: More Elementary Matrices

*Week 8* | *UCSB 2013*

In this lecture, we continue our discussion of the **elementary** matrices.

## 1 Elementary Matrices: Recap

In our last class, we defined the elementary matrices

$$E_{\text{multiply } k \text{ by } \lambda} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \lambda & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}, \quad E_{\text{switch } k \text{ and } l} = \begin{bmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 1 \end{bmatrix},$$

$$E_{\text{add } \lambda k \text{ to } l} = \begin{bmatrix} 1 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & \ldots & \lambda & 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

From there, we proved the following theorem, that explains what happens when we multiply other matrices by these objects:

**Theorem 1.** *Take any $n \times n$ matrix $A$. Suppose that we are looking at the composition $E \circ A$, where $E$ is one of our elementary matrices. Then, we have the following three possible situations:*

- *if $E = E_{\text{multiply entry } k \text{ by } \lambda}$, then $E \circ A$ would be the matrix $A$ with its $k$-th row multipled by $\lambda$.*
- *if $E = E_{\text{switch entry } k \text{ and entry } l}$, then $E \circ A$ would be the matrix $A$ with its $k$-th and $l$-th rows swapped, and*
- *if $E = E_{\text{add } \lambda \text{ copies of entry } k \text{ to entry } l}$, then $E \circ A$ would be the matrix $A$ with $\lambda$ copies of its $k$-th row added to its $l$-th row.*

## 2 Creating Arbitrary Matrices Using Elementary Matrices

In today's talk, we explain part of why we care about elementary matrices: they give us a way to create **any** $n \times n$ matrix!

**Theorem.** Let $A$ be a $n \times n$ matrix. Then there is some string of elementary matrices $E_1, \ldots E_k$ such that

$$A = E_1 \circ \ldots \circ E_k$$

Before proving this theorem, we work an example, to give an idea for how the proof will go:

**Example.** Take the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 5 & 8 \\ 13 & 21 & 34 \end{bmatrix}.$$

Write it as a product of elementary matrices.

*Proof.* Our plan of attack is the following: we will start with some matrix $B$, and apply elementary matrices to it until it becomes the desired matrix $A$. Specifically:

1. We will start with $B$ equal to the identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2. From here, we will use the theorem we proved in class, that tells us how our elementary matrices alter other matrices! Specifically, starting with the first row and working our way down, we will perform operations on the rows of B, that will row-by-row turn $B$ into $A$.

   (a) Specifically: to make sure that our matrix has the same first row as $A$, we will find a linear combination of the three rows of $B$ that yields the first row of $A$. We will then make our matrix have this first row by **applying the elementary matrices that correspond to that linear combination**! In other words: suppose that we have a linear combination

   $$\alpha \vec{b_{r_1}} + \beta \vec{b_{r_2}} + \gamma \vec{b_{r_3}} = \vec{a_{r_1}}.$$

   Then, if we calculate

   $$\overbrace{\begin{bmatrix} 1 & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{add } \gamma \text{ copies of} \\ r_3 \text{ to } r_1}} \cdot \overbrace{\begin{bmatrix} 1 & \beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{add } \beta \text{ copies of} \\ r_2 \text{ to } r_1}} \cdot \overbrace{\begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{multiply row } r_1 \\ \text{by } \alpha}} \cdot B,$$

   we will get the matrix where the first row of $B$ now contains

- $\alpha$ copies of what used to be its first row, plus
- $\beta$ copies of its second row, plus
- $\gamma$ copies of its third row!

But we said that this combination is $\vec{a_{r_1}}$ – so our matrix now has the same first row as $\vec{a_{r_1}}$!

(Notice that we applied the "multiply" matrix first! This is because we want to get $\alpha\vec{b_{r_1}} + \beta\vec{b_{r_2}} + \gamma\vec{b_{r_3}}$ in the first row. If we were to add $\beta$ copies of the second row to the first row, and then multiply the first row by $\alpha$, we'd accidentally get $\beta\alpha\vec{b_{r_2}}$ copies of the second row in the first row, which is not what we want.)

(b) We then repeat this process on $B$'s second row! Specifically, we will find a linear combination

$$\alpha\vec{b_{r_1}} + \beta\vec{b_{r_2}} + \gamma\vec{b_{r_3}} = \vec{a_{r_2}}.$$

Then, if we calculate

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{add } \alpha \text{ copies of} \\ r_1 \text{ to } r_2}} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{add } \gamma \text{ copies of} \\ r_3 \text{ to } r_2}} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{multiply row } r_2 \\ \text{by } \beta}} \cdot B,$$

we again get the matrix where the second row of $B$ now contains

- $\beta$ copies of what used to be its first row, plus
- $\alpha$ copies of its first row, plus
- $\gamma$ copies of its third row!

Again, this is $\vec{a_{r_2}}$. Yay!

(c) We do this one more time. Again, find a linear combination

$$\alpha\vec{b_{r_1}} + \beta\vec{b_{r_2}} + \gamma\vec{b_{r_3}} = \vec{a_{r_3}}.$$

Then, if we calculate

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{bmatrix}}^{\substack{\text{add } \alpha \text{ copies of} \\ r_1 \text{ to } r_3}} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \beta & 1 \end{bmatrix}}^{\substack{\text{add } \beta \text{ copies of} \\ r_2 \text{ to } r_3}} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma \end{bmatrix}}^{\substack{\text{multiply row } r_3 \\ \text{by } \gamma}} \cdot B,$$

we again get the matrix where the third row of $B$ now contains

- $\gamma$ copies of what used to be its third row, plus
- $\alpha$ copies of its first row, plus
- $\beta$ copies of its second row!

This is $\vec{a_{r3}}$, again using similar logic to before. In other words, we've successfully turned our matrix $B$ into $A$! If we simply write down all of the nine elementary matrices we used on the way, this gives us a way to write our matrix $A$ as the product of these nine elementary matrices with the identity matrix (which is itself an elementary matrix!)

Ok. With this plan established, we just have to do it for

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 5 & 8 \\ 13 & 21 & 34 \end{bmatrix}.$$

We start with $B$ equal to the identity matrix, and we try to make its first row equal to $A$'s first row. Here, we want $\alpha, \beta, \gamma$ such that

$$\alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1) = (1,1,2);$$

in other words $\alpha = 1, \beta = 1, \gamma = 2$.

So we have

$$\overbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{add 2 copies of} \\ r_3 \text{ to } r_1}} \cdot \overbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{add 1 copies of} \\ r_2 \text{ to } r_1}} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{multiply row } r_1 \\ \text{by } 1}} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\text{the matrix } B} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This rightmost matrix is the new $B$.

Now, we do it again for the second row! I.e. we want $\alpha, \beta, \gamma$ such that

$$\alpha(1,1,2) + \beta(0,1,0) + \gamma(0,0,1) = (3,5,8);$$

in other words $\alpha = 3, \beta = 2, \gamma = 2$.

So we have

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{add 3 copies of} \\ r_1 \text{ to } r_2}} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{add 2 copies of} \\ r_3 \text{ to } r_2}} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{multiply row } r_2 \\ \text{by } 2}} \cdot \overbrace{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\text{the matrix } B} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 5 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

Again, this rightmost matrix is the new $B$.

Do it again for the third row! I.e. find $\alpha, \beta, \gamma$ such that

$$\alpha(1,1,2) + \beta(3,5,8) + \gamma(0,0,1) = (13,21,34);$$

in other words $\alpha = 1, \beta = 4, \gamma = 0$.

This gives us

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}^{\substack{\text{add 1 copies of} \\ r_1 \text{ to } r_3}} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}}^{\substack{\text{add 4 copies of} \\ r_2 \text{ to } r_3}} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{\substack{\text{multiply row } r_3 \\ \text{by } 0}} \cdot \overbrace{\begin{bmatrix} 1 & 1 & 2 \\ 3 & 5 & 8 \\ 0 & 0 & 1 \end{bmatrix}}^{\text{the matrix } B} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 5 & 8 \\ 13 & 21 & 34 \end{bmatrix}.$$

Win! We've therefore written $A$ as the product of elementary matrices! Specifically, we've written

$$\overbrace{\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}}^{\substack{\text{add 1 copies of}\\ r_1 \text{ to } r_3}} \cdot \overbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&4&1\end{bmatrix}}^{\substack{\text{add 4 copies of}\\ r_2 \text{ to } r_3}} \cdot \overbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}}^{\substack{\text{multiply row } r_3\\ \text{by } 0}} \cdot \overbrace{\begin{bmatrix}1&0&0\\3&1&0\\0&0&1\end{bmatrix}}^{\substack{\text{add 3 copies of}\\ r_1 \text{ to } r_2}} \cdot \overbrace{\begin{bmatrix}1&0&0\\0&1&2\\0&0&1\end{bmatrix}}^{\substack{\text{add 2 copies of}\\ r_3 \text{ to } r_2}} \cdot \overbrace{\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}}^{\substack{\text{multiply row } r_2\\ \text{by } 2}}$$

$$\cdot \overbrace{\begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}}^{\substack{\text{add 2 copies of}\\ r_3 \text{ to } r_1}} \cdot \overbrace{\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}}^{\substack{\text{add 1 copies of}\\ r_2 \text{ to } r_1}} \cdot \overbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}^{\substack{\text{multiply row } r_1\\ \text{by } 1}} \cdot \overbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}^{\text{the identity matrix}} .$$

□

This method doesn't always blindly work, however: consider the following matrix!

$$A = \begin{bmatrix}0&1\\1&0\end{bmatrix}$$

If we were to simply apply the method above, we'd start with

$$B = \begin{bmatrix}1&0\\0&1\end{bmatrix}.$$

From there, we would try to combine the two rows of $B$ to get the first row of $A$:

$$\alpha(1,0) + \beta(0,1) = (0,1) \to \alpha = 0, \beta = 1.$$

This would have us perform the following two steps:

$$\overbrace{\begin{bmatrix}1&1\\0&1\end{bmatrix}}^{\substack{\text{add 1 copies of}\\ r_2 \text{ to } r_1}} \cdot \overbrace{\begin{bmatrix}0&0\\0&1\end{bmatrix}}^{\substack{\text{multiply row } r_1\\ \text{by } 0}} \cdot \overbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}}^{\text{the matrix } B} = \begin{bmatrix}0&1\\0&1\end{bmatrix}.$$

From here, however, we're stuck! There's no way to keep going.

The thing that happened here is that we turned $B$ from a matrix with two linearly independent rows into a matrix whose rows are linearly dependent! Because we did this, there was no way to continue: all possible multiples of our rows were things of the form $(0, x)$, and we can't make $(1, 0)$ out of such vectors.

However, we can get around this by using the **swap** elementary matrices! In the example above, the matrix $A$ is itself a swap matrix: so it's already an elementary matrix!

In general, by using these swaps we can avoid the situation above, where we made the rows linearly dependent "too soon." In practice, you can just do this by feel, but if you want a rigorous approach, use the following theorem!

**Theorem.** Let $A$ be an arbitrary $n \times n$ matrix. Then we can write $A$ as the product of elementary matrices.

*Proof.* To do this process, first do the following:

1. Take the collection $R$ of all of $A$'s rows.

2. If this set is linearly independent, you're done!

3. Otherwise, there is some row that shows up in this collection that is a combination of the other rows. Get rid of that row, and return to (2).

This creates a subset $R'$ of $A$'s rows that is linearly independent. Furthermore, it creates a subset from which we can create **any** of $A$'s rows, even the ones we got rid of! This is because we only got rid of rows that were linearly dependent on the earlier ones; i.e. we only got rid of rows that we can make with the rows we kept!

So: all we need to do now is make $B$ into a matrix that has all of the rows in this subset $R'$! If we can do this, then we can just do the following:

- Multiply all of the other rows in $B$ by zero.

- Now, using each all-zero row as an empty slot, create each of the rows from $A$ that we don't have by combining the rows from $R'$. We can do this because all of the remaining rows in $A$ were linear combinations of the $R'$ rows!

- Finally, rearrange the rows using swaps so that our matrix **is** $A$ (and not just a matrix with the same rows, but in some different order.)

This is our plan! We execute the plan as below:

1. We start with $B$ equal to the $n \times n$ identity matrix. Note that $B$'s rows span all of $\mathbb{R}^n$

2. If all of the rows in $R'$ currently occur as rows of $B$, stop!

3. Otherwise, there is a row $\vec{a_r}$ in $R'$ that is not currently a row in $B$.

4. If the rows of $B$ span $\mathbb{R}$, then specifically there is a combination of the rows of $B$ that yields $\vec{a_r}$.

5. Furthermore, this vector is not just a combination of rows in $R'$, because $R'$ is a linearly independent set. Therefore, in any linear combination of $B$'s rows that creates $\vec{a_r}$, there is some row of $B$ that is not one of the $R'$ rows that's used in creating $\vec{a_r}$.

6. So: take the linear combination

$$a_1 \vec{b_{r_1}} + \ldots a_n \vec{b_{r_n}} = \vec{a_r},$$

   and let $\vec{b_{r_k}}$ denote the row that occurs above that's not one of the $R'$ rows and that has $a_k \neq 0$.

7. Take $B$, and multiply it by

$$\overbrace{\underbrace{\begin{bmatrix} & \vdots & \\ \dots & a_1 & \dots \\ & \vdots & \end{bmatrix}}^{\substack{\text{add } a_1 \text{ copies of} \\ r_1 \text{ to } r_k}} \cdot \overbrace{\begin{bmatrix} & \vdots & \\ \dots & a_2 & \dots \\ & \vdots & \end{bmatrix}}^{\substack{\text{add } a_2 \text{ copies of} \\ r_2 \text{ to } r_k}} \cdot \ldots \cdot \overbrace{\begin{bmatrix} & \vdots & \\ \dots & a_n & \dots \\ & \vdots & \end{bmatrix}}^{\substack{\text{add } a_n \text{ copies of} \\ r_2 \text{ to } r_k}}}^{\text{all of the values } a_i, \text{ with } i \neq k} \cdot \overbrace{\begin{bmatrix} & \vdots & \\ \dots & a_k & \dots \\ & \vdots & \end{bmatrix}}^{\substack{\text{multiply row } r_k \\ \text{by } a_k}} \cdot B$$

This takes the $k$-th row of $B$ and fills it with the linear combination that creates $\vec{a_r}$! So this means that the row $\vec{a_r}$ is now in $B$.

8. Also, notice that the rows of $B$ all still span $\mathbb{R}^n$! This is because

$$a_1\vec{b_{r_1}} + \dots a_n\vec{b_{r_n}} = \vec{a_r}$$
$$\Rightarrow \vec{b_{r_k}} = \frac{1}{a_k}\left(\underbrace{a_1\vec{b_{r_1}} + \dots a_n\vec{b_{r_n}}}_{\text{terms that aren't } a_k\vec{b_{r_k}}} + \vec{a_r}\right).$$

Therefore, we have that the old $k$-th row $\vec{b_{r_k}}$ is in the span of the new $B$'s rows! As well, because none of the other rows changed, those rows are all still in the span as well. Therefore, because the new $B$'s rows contain the old $B$'s rows in their span, they must span $\mathbb{R}^n$!

9. Go to (2), and repeat this process!

The result of this process is a matrix $B$ that contains all of the rows in $R'$, which is what we wanted (because we can make $A$ out of this!) So we're done. □

To illustrate this argument, we run another example:

**Example.** Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 4 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Write $A$ as a product of elementary matrices.

*Proof.* We start, as directed in the proof, by finding a subset of $A$'s rows that is linearly independent. We can tell at the start that the collection of all rows is not linearly independent, because

$$1(0,1,2) + 1(4,-1,0) - 2(2,0,1) = (0,0,0).$$

However, we also have that the pair

$$(0,1,2), (2,0,1)$$

**is** linearly independent, because

$$\alpha(0,1,2) + \beta(2,0,1) = (0,0,0) \Rightarrow \alpha, \beta = 0,$$

and that these two vectors contain the third in their span.

So the set $R'$ from our discussion above is just these two vectors!

Set $B$ equal to the $3 \times 3$ identity matrix. We start by picking a vector from $R'$ – let's choose $\vec{a_r} = (0,1,2)$.

We want to multiply $B$ by elementary matrices so that it has $(0,1,2)$ as one of its rows. To do this, we first write $(0,1,2)$ as a combination of $B$'s rows:

$$0(1,0,0) + 1(0,1,0) + 2(0,0,1) = (0,1,2).$$

We now pick a row from $B$ whose coefficient above is nonzero, and that isn't a row in $R'$. For example, the coefficient of the second row above is 1, and the second row $(0,1,0)$ is not in $R'$: so we can pick the second row.

We now turn the second row into this $\vec{a_r} = (0,1,2)$, by using the linear combination we have for $(0,1,2)$ above:

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{add 2 copies of} \\ r_3 \text{ to } r_2}} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{add 0 copies of} \\ r_1 \text{ to } r_2}} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{multiply row } r_2 \\ \text{by } 1}} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\text{the matrix } B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Success! We repeat this. We choose another row from $R'$, specifically $\vec{a_r} = (2,0,1)$. We write $(2,0,1)$ as a combination of $B$'s rows:

$$2(1,0,0) + 0(0,1,2) + 1(0,0,1) = (2,0,1).$$

We now pick a row from $B$ whose coefficient above is nonzero, and that isn't a row in $R'$; for example, the first row works here.

We now turn the first row into this $\vec{a_r} = (2,0,1)$, by using the linear combination we have for $(2,0,1)$ above:

$$\overbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{add 1 copies of} \\ r_3 \text{ to } r_1}} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{add 0 copies of} \\ r_2 \text{ to } r_1}} \cdot \overbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{multiply row } r_1 \\ \text{by } 2}} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}^{\text{the matrix } B} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

We are now out of rows of $R'$! This brings us to the second stage of our proof: multiply all of the remaining rows that aren't $R'$ rows by 0.

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}^{\substack{\text{multiply row } r_3 \\ \text{by } 0}} \cdot \overbrace{\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}^{\text{the matrix } B} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now we are at the last stage of our proof: combine the $R'$ rows to create whatever rows in $A$ are left, in these "blank" all-zero rows!

Specifically, we take the one row of $A$ that's left: $(4, -1, 0)$. As we noted before, we can write

$$(4, -1, 0) = 2(2, 0, 1) - 1(0, 1, 2).$$

Therefore, we have

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}^{\substack{\text{add 2 copies of} \\ r_1 \text{ to } r_3}} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}^{\substack{\text{add } -1 \text{ copies of} \\ r_2 \text{ to } r_3}} \cdot \overbrace{\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}^{\text{the matrix } B} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 4 & -1 & 0 \end{bmatrix}.$$

So we have a matrix with the same rows as $A$! Finally, we just shuffle the rows of $B$ to get $A$ itself:

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}^{\substack{\text{switch rows} \\ r_3 \text{ and } r_2}} \cdot \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\substack{\text{switch rows} \\ r_2 \text{ and } r_1}} \cdot \overbrace{\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 4 & -1 & 0 \end{bmatrix}}^{\text{the matrix } B} = \begin{bmatrix} 0 & 1 & 2 \\ 4 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = A.$$

Win! $\square$