| Math 108a | Professor: Padraic Bartlett |
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# Lecture 16: Elementary Matrices

| *Week 8* | *UCSB 2013* |
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Last week, we introduced the idea of **matrices**. In this lecture, we introduce a series of special kinds of matrices: the **elementary** matrices.

## 1 Elementary Matrices: Definitions

**Definition.** The first matrix, $E_{\text{multiply entry k by }\lambda}$, is the matrix corresponding to the linear map that multiplies its $k$-th coördinate by $\lambda$ and does not change any of the others. Specifically, it's the matrix corresponding to the linear map

$$(x_1, x_2 \ldots x_n) \mapsto (x_1, x_2, \ldots x_{k-1}, \lambda x_k, x_{k+1}, \ldots x_n).$$

If we plug in the standard basis vectors $\vec{e_1}, \ldots \vec{e_n}$ into this linear map, we can see that we have

$$\begin{aligned}
\vec{e_1} &= (1, 0, 0 \ldots 0) \mapsto (1, 0, 0 \ldots 0),\\
\vec{e_2} &= (0, 1, 0 \ldots 0) \mapsto (0, 1, 0 \ldots 0),\\
&\vdots\\
\vec{e_k} &= (0 \ldots \overbrace{1}^{k\text{-th coordinate}} \ldots 0) \mapsto (0 \ldots \overbrace{\lambda}^{k\text{-th coordinate}} \ldots 0),\\
&\vdots\\
\vec{e_n} &= (0 \ldots 0, 1) \mapsto (0 \ldots 0, 1).
\end{aligned}$$

If we use these outputs as our columns, we can see that our linear map corresponds to the following matrix:

$$E_{\text{multiply entry } k \text{ by } \lambda} = \begin{bmatrix}
1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0\\
0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0\\
0 & 0 & \ldots & 0 & \lambda & 0 & \ldots & 0\\
0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1
\end{bmatrix}$$

This matrix has 1's down its diagonal and 0's elsewhere, with an exception for the value at $(k, k)$, which is $\lambda$.

The second matrix, $E_{\text{switch entry } k \text{ and entry } l}$, corresponds to the linear map that swaps its $k$-th coördinate with its $l$-th coördinate, and does not change any of the others. Specifically, it's the matrix corresponding to the linear map

$$(x_1, x_2 \ldots x_n) \mapsto (x_1, x_2, \ldots x_{k-1}, x_l, x_{k+1}, \ldots x_{l-1}, x_k, x_{l+1}, \ldots x_n).$$

If we plug in the standard basis vectors $\vec{e_1}, \ldots \vec{e_n}$ into this linear map, we can see that we have

$$\begin{aligned}
\vec{e_1} &= (1, 0, 0 \ldots 0) \mapsto (1, 0, 0 \ldots 0), \\
\vec{e_2} &= (0, 1, 0 \ldots 0) \mapsto (0, 1, 0 \ldots 0), \\
&\vdots \\
\vec{e_k} &= (0 \ldots \overbrace{1}^{k\text{-th coordinate}} \ldots 0) \mapsto (0 \ldots \overbrace{1}^{l\text{-th coordinate}} \ldots 0), \\
&\vdots \\
\vec{e_l} &= (0 \ldots \overbrace{1}^{l\text{-th coordinate}} \ldots 0) \mapsto (0 \ldots \overbrace{1}^{k\text{-th coordinate}} \ldots 0), \\
&\vdots \\
\vec{e_n} &= (0 \ldots 0, 1) \mapsto (0 \ldots 0, 1).
\end{aligned}$$

If we use these outputs as our columns, we can see that our linear map corresponds to the following matrix:

$$E_{\text{switch entry } k \text{ and entry } l} = \begin{bmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 1 \end{bmatrix}$$

You can create this matrix by starting with a matrix with 1's down its diagonal and 0's elsewhere, and switching the $k$-th and $l$-th columns.

Finally, the third matrix, $E_{\text{add } \lambda \text{ copies of entry } k \text{ to entry } l}$, for $k \neq l$, corresponds to the linear map that adds $\lambda$ copies of its $k$-th coördinate to its $l$-th coördinate and does not change any of the others. Specifically, it's the matrix corresponding to the linear map

$$(x_1, x_2 \ldots x_n) \mapsto (x_1, x_2, \ldots x_{l-1}, \lambda x_k + x_l, x_{l+1}, \ldots x_n).$$

If we plug in the standard basis vectors $\vec{e_1}, \ldots \vec{e_n}$ into this linear map, we can see that we have

$$\begin{aligned}
\vec{e_1} &= (1,0,0\dots 0) \mapsto (1,0,0\dots 0),\\
\vec{e_2} &= (0,1,0\dots 0) \mapsto (0,1,0\dots 0),\\
&\vdots\\
\vec{e_l} &= (0\dots \overbrace{1}^{l\text{-th coordinate}} \dots 0) \mapsto (0\dots \overbrace{\lambda}^{k\text{-th coordinate}} \dots \overbrace{1}^{l\text{-th coordinate}} \dots 0),\\
&\vdots\\
\vec{e_n} &= (0\dots 0,1) \mapsto (0\dots 0,1).
\end{aligned}$$

If we use these outputs as our columns, we can see that our linear map corresponds to the following matrix:

$$E_{\text{add } \lambda \text{ copies of entry } k \text{ to entry } l} = \begin{bmatrix}
1 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0\\
0 & \dots & 0 & 1 & \dots & 0 & 0 & \dots & 0\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & \dots & 0 & 0 & \dots & 1 & 0 & \dots & 0\\
0 & \dots & \lambda & 0 & \dots & 0 & 1 & \dots & 0\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1
\end{bmatrix}$$

This matrix has 1's down its diagonal and 0's elsewhere, with an exception for the value in row $l$, column $k$, which is $\lambda$.

These three matrices are called the **elementary** matrices. They're incredibly cool, and we're going to study them in these lecture notes.

## 2 Elementary Matrices: What They Do

The first thing we want to talk about is what these matrices **do**! Specifically, take any $n \times n$ matrix $A$. What is the matrix corresponding to $E_{\text{multiply entry k by } \lambda} \circ A$? What do the other elementary matrices do to $A$?

We study this in the following theorem:

**Theorem 1.** *Take any $n \times n$ matrix $A$. Suppose that we are looking at the composition $E \circ A$, where $E$ is one of our elementary matrices. Then, we have the following three possible situations:*

- *if $E = E_{\text{multiply entry } k \text{ by } \lambda}$, then $E \circ A$ would be the matrix $A$ with its $k$-th row multiplied by $\lambda$.*

- *if $E = E_{\text{switch entry } k \text{ and entry } l}$, then $E \circ A$ would be the matrix $A$ with its $k$-th and $l$-th rows swapped, and*
- *if $E = E_{\text{add } \lambda \text{ copies of entry } k \text{ to entry } l}$, then $E \circ A$ would be the matrix $A$ with $\lambda$ copies of its $k$-th row added to its $l$-th row.*

*Proof.* To prove these claims, we repeatedly use the following result from last week, that told us how to "compose" or "multiply" two matrices together:

**Theorem.** Take any pair of linear maps $A : \mathbb{R}^n \to \mathbb{R}^m, B : \mathbb{R}^m \to \mathbb{R}^k$ with associated matrices

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}, B = \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,m} \\ b_{2,1} & b_{2,2} & \dots & b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k,1} & b_{k,2} & \dots & b_{k,m} \end{bmatrix}.$$

Look at the linear map given by the composition of these two maps: i.e. consider the linear map $B \circ A : \mathbb{R}^n \to \mathbb{R}^k$. Denote the row vectors of $B$ as $\vec{b_{r_i}}$'s and the column vectors of $A$ as $\vec{a_{c_j}}$'s. We claim that this linear map corresponds to the $k \times n$ matrix

$$\begin{bmatrix} \vec{b_{r_1}} \cdot \vec{a_{c_1}} & \vec{b_{r_1}} \cdot \vec{a_{c_2}} & \dots & \vec{b_{r_1}} \cdot \vec{a_{c_n}} \\ \vec{b_{r_2}} \cdot \vec{a_{c_1}} & \vec{b_{r_2}} \cdot \vec{a_{c_2}} & \dots & \vec{b_{r_2}} \cdot \vec{a_{c_n}} \\ \dots & \dots & \ddots & \dots \\ \vec{b_{r_k}} \cdot \vec{a_{c_1}} & \vec{b_{r_k}} \cdot \vec{a_{c_2}} & \dots & \vec{b_{r_k}} \cdot \vec{a_{c_n}} \end{bmatrix}.$$

In other words, to get the matrix given by composing two matrices, we simply dot the rows of the first matrix with the columns of the second matrix in the manner described above.

Given this result, we simply calculate $E \circ A$ for each of the three cases we've described above.

To start, take any $n \times n$ matrix $A$, row $k$ and constant $\lambda$, and examine the product

$$E_{\text{multiply entry } k \text{ by } \lambda} \circ A$$

$$= \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \lambda & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \circ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \dots & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & \dots & a_{4n} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & \dots & a_{5n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & a_{n5} & \dots & a_{nn} \end{bmatrix}.$$

What do entries in the resulting matrix look like? Well, there are two cases:

- in the location $(i, j)$, for any $i \neq k$ and any $j$, we know that the entry there is just the dot product of $E$'s $i$-th row and $A$'s $j$-th column: i.e.

$$\text{entry } (i,j) = (0,\ldots,1,\ldots 0)\cdot(a_{1j},\ldots a_{nj}) = a_{ij},$$

  because the 1 in the $i$-th row of $E$ is in the $i$-th place.

- in the location $(k, j)$, for any $j$, we know that the entry there is just the dot product of $E$'s $k$-th row and $A$'s $j$-th column: i.e.

$$\text{entry } (k,j) = (0,\ldots,\lambda,\ldots 0)\cdot(a_{1j},\ldots a_{nj}) = \lambda_{kj},$$

  because the $\lambda$ in the $k$-th row of $E$ is in the $k$-th place.

By inspection, this matrix is precisely

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & a_{k-1,4} & a_{k-1,5} & \ldots & a_{k-1,n} \\ \lambda a_{k1} & \lambda a_{k2} & \lambda a_{k3} & \lambda a_{k4} & \lambda a_{k5} & \ldots & \lambda a_{kn} \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & a_{k+1,4} & a_{k+1,5} & \ldots & a_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & a_{n5} & \ldots & a_{nn} \end{bmatrix}.$$

So this elementary matrix works as claimed.

The proofs for the other two elementary matrices are similar. For the matrix $E_{\text{switch entry } k \text{ and entry } l}$, we again examine the product $E \circ A$:

$$E_{\text{switch entry } k \text{ and entry } l} \circ A$$
$$= \begin{bmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 1 \end{bmatrix} \circ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \ldots & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & \ldots & a_{4n} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & \ldots & a_{5n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & a_{n5} & \ldots & a_{nn} \end{bmatrix}.$$

Again, what do entries in the resulting matrix look like? In this situation, there are three cases:

- In the location $(i, j)$, for any $i \neq k, l$ and any $j$, we know that the entry there is just the dot product of $E$'s $i$-th row and $A$'s $j$-th column: i.e.

$$\text{entry } (i,j) = (0,\ldots,1,\ldots 0)\cdot(a_{1j},\ldots a_{nj}) = a_{ij},$$

  because the 1 in the $i$-th row of $E$ is in the $i$-th place.

- In the location $(k, j)$, for any $j$, we know that the entry there is just the dot product of $E$'s $k$-th row and $A$'s $j$-th column: i.e.

$$\text{entry } (k, j) = (0, \ldots, 1, \ldots 0) \cdot (a_{1j}, \ldots a_{nj}) = a_{lj},$$

  because the 1 in the $k$-th row of $E$ is in the $l$-th place.

- In the location $(l, j)$, for any $j$, we know that the entry there is just the dot product of $E$'s $l$-th row and $A$'s $j$-th column: i.e.

$$\text{entry } (l, j) = (0, \ldots, 1, \ldots 0) \cdot (a_{1j}, \ldots a_{nj}) = a_{kj},$$

  because the 1 in the $l$-th row of $E$ is in the $k$-th place.

By inspection, this matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & a_{k-1,4} & a_{k-1,5} & \ldots & a_{k-1,n} \\ a_{l1} & a_{l2} & a_{l3} & a_{l4} & a_{l5} & \ldots & a_{ln} \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & a_{k+1,4} & a_{k+1,5} & \ldots & a_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{l-1,1} & a_{l-1,2} & a_{l-1,3} & a_{l-1,4} & a_{l-1,5} & \ldots & a_{l-1,n} \\ a_{k1} & a_{k2} & a_{k3} & a_{k4} & a_{k5} & \ldots & a_{kn} \\ a_{l+1,1} & a_{l+1,2} & a_{l+1,3} & a_{l+1,4} & a_{l+1,5} & \ldots & a_{l+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & a_{n5} & \ldots & a_{nn} \end{bmatrix}.$$

This is $A$ with its $k$-th and $l$-th rows swapped, as claimed.

Finally, we turn to $E_{\text{add } \lambda \text{ copies of entry } k \text{ to entry } k}$, and again look at $E \circ A$:

$$E_{\text{add } \lambda \text{ copies of entry } k \text{ to entry } l} \circ A$$
$$= \begin{bmatrix} 1 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & \ldots & \lambda & 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 1 \end{bmatrix} \circ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \ldots & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & \ldots & a_{4n} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & \ldots & a_{5n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & a_{n5} & \ldots & a_{nn} \end{bmatrix}.$$

Again, what do entries in the resulting matrix look like? In this situation, there are just two last cases:

- In the location $(i, j)$, for any $i \neq l$ and any $j$, we know that the entry there is just the dot product of $E$'s $i$-th row and $A$'s $j$-th column: i.e.

$$\text{entry } (i, j) = (0, \ldots, 1, \ldots 0) \cdot (a_{1j}, \ldots a_{nj}) = a_{ij},$$

because the 1 in the $i$-th row of $E$ is in the $i$-th place.

- In the location $(l, j)$, for any $j$, we know that the entry there is just the dot product of $E$'s $k$-th row and $A$'s $j$-th column: i.e.

$$\text{entry } (k, j) = (0, \ldots, 0, \lambda, 0, \ldots, 0, 1, 0, \ldots 0) \cdot (a_{1j}, \ldots a_{nj}) = \lambda a_{kj} + a_{lj},$$

because the $\lambda$ in the $l$-th row of $E$ is in the $k$-th place, and the 1 is in the $l$-th place.

By inspection, this matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{l-1,1} & a_{l-1,2} & a_{l-1,3} & a_{l-1,4} & a_{l-1,5} & \ldots & a_{l-1,n} \\ \lambda a_{k1} + a_{l1} & \lambda a_{k2} + a_{l2} & \lambda a_{k3} + a_{l3} & \lambda a_{k4} + a_{l4} & \lambda a_{k5} + a_{l5} & \ldots & \lambda a_{kn} + a_{ln} \\ a_{l+1,1} & a_{l+1,2} & a_{l+1,3} & a_{l+1,4} & a_{l+1,5} & \ldots & a_{l+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & a_{n5} & \ldots & a_{nn} \end{bmatrix}.$$

This is $A$ with $\lambda$ times its $k$-th row added to its $l$-th row, as claimed. $\square$