Math 108a	Professor: Padraic Bartlett

Lecture 14: The Rank-Nullity Theorem

Week 6	*UCSB 2013*

In today's talk, the last before we introduce the concept of **matrices**, we prove what is arguably the strongest theorem we've seen thus far this quarter – the **rank-nullity** theorem!

1 The Rank-Nullity Theorem: What It Is

The **rank-nullity** theorem is the following result:

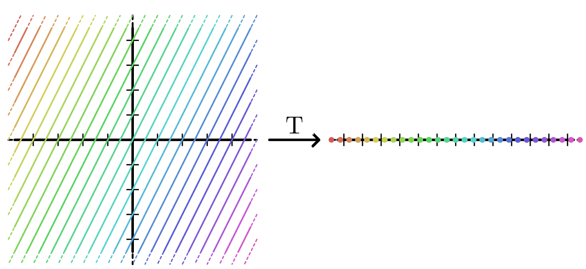
Theorem. Let U, V be a pair of finite-dimensional vector spaces, and let $T : U \to V$ be a linear map. Then the following equation holds:

$$\text{dimension}(\text{null}(T)) + \text{dimension}(\text{range}(T)) = \text{dimension}(U).$$

Though we have never mentioned this theorem in class before, we have proved that it holds in a number of specific situations in earlier classes! For example, in lecture 11 where we were describing why the null space was interesting, we considered the linear map $T : \mathbb{R}^2 \to \mathbb{R}, T(x, y) = 2x - y$. Using this map, we made the observations that

1. the null space of T is the line $\{(x, 2x) \mid x \in \mathbb{R}\}$, and
2. for any $a \in \mathbb{R}$, $T^{-1}(a)$ is just a copy of this line shifted by some constant: i.e. $T^{-1}(a) = \{(0, -a) + \vec{n} \mid \vec{n} \in \text{null}(T)\}$.

We illustrated this situation before with the following picture:



Graphically, what we did here is decompose the **domain** of our linear map, $\mathbb{R}^2$, into "range(T)-many" copies of the null space. If we look at the dimensions of the objects we studied here, we said that we could take a two-dimensional object (the **domain**) and break it into copies of this one dimensional object (the **null space**), with as many copies of this one-dimensional object as we have $T^{-1}(a)$'s (i.e. elements in the **range**.)

In general, we said in this class that we could always break the domain into copies of the null space, with as many copies as we have sets $T^{-1}(a)$'s, i.e. elements a in the range! This should lead you to believe that the rank-nullity theorem is true: it certainly seems like it should always hold, given our work in that earlier lecture.

However, this might not feel a lot like a proper "proof" to some of you: there's a pretty picture here, but where is the rigor? How am I turning this decomposition into an argument about dimension (which is a statement about bases, which we're not saying anything about here?)

The point of this lecture is to take the above intuition, and turn it into a proper argument. We do so as follows:

2 The Proof: Some Useful Lemmas

First, recall the following theorem, that we proved over the course of two days earlier in the class (see lecture 6):

Theorem. The idea of "dimension" is well defined. In other words: suppose that U is a vector space with two different bases B_1, B_2 containing finitely many elements each. Then there are as many elements in B_1 as there are in B_2.

We will need this theorem to prove the rank-nullity theorem. As well, we will also need the following:

Theorem. Suppose that U is a n-dimensional vector space with basis B, and that S is a subspace of U. Then S is also finite dimensional, and in particular has dimension no greater than n.

Proof. We prove this statement by contradiction. Suppose that S is a set with dimension greater than n. Using this fact, create a set T of $n+1$ linearly independent vectors in S as follows. At first, simply pick any nonzero vector $\vec{v} \in S$, and put $\vec{v} \in T$. Then, repeat the following process

1. Look at T. If it has no more than n vectors in it, then it cannot span S, because S has dimension greater than n. So there is some vector $\vec{w} \in S$ that is not in the span of T.

2. Put $\vec{w}$ in T. Notice that if T was linearly independent before we added $\vec{w}$, it is still linearly independent, because $\vec{w}$ was not in the span of T.

So: we now have a set of $n+1$ linearly independent vectors in S, which means in particular we have a set of $n+1$ linearly independent vectors in U! This is a problem, because U is n-dimensional. To see why, notice that we can"grow" this set T of $n+1$ linearly independent vectors into a basis for U as follows. Start with T. If T spans U, stop. Otherwise, repeat the following process until we get a linearly independent set that spans U:

1. If T does not span U, then there is some element $\vec{b}$ in the basis B of U that is not in the span of T (because otherwise, if T contained all of a basis for U in its span, it would be forced to contain all of U itself in its span!).

2. Add $\vec{b}$ to T. Again, notice that if T was linearly independent before we added $\vec{b}$, it is still linearly independent, because $\vec{b}$ was not in the span of T.

Eventually, this process will stop, as after n steps your set T will at the least contain **all** of the elements of B, which is a basis for U!

So this creates a linearly independent set that spans U: i.e. a basis! And in particular a basis with at least $n+1$ elements, because T started with $n+1$ elements and had more things potentially added to it later.

However, B is a basis for U with n elements. We've proven that dimension is well-defined: i.e. that a vector space cannot have two different bases with different sizes! Therefore, this is impossible, and we have a contradiction. Consequently, our assumption must be false: in other words, S must have dimension no greater than n. □

3 The Main Proof

With these tools set up, we proceed with the main proof:

Theorem. Let U, V be a pair of finite-dimensional vector spaces, and let $T : U \to V$ be a linear map. Then the following equation holds:

$$\text{dimension}(\text{null}(T)) + \text{dimension}(\text{range}(T)) = \text{dimension}(U).$$

Proof. We prove this as follows: we will create a basis for U with as many elements as $\text{dimension}(\text{null}(T)) + \text{dimension}(\text{range}(T))$, which will clearly demonstrate the above equality. We do this as follows:

1. By using Theorem 2, we can see that the null space of T is finite dimensional. Take some basis N_B for the null space of T.

2. Now: let B be a basis for U, and R_B be some set that is empty for now. If N_B is also a basis for U, stop. Otherwise, repeat the following process:

   (a) Look at $N_B \cup R_B$. If it is a basis for U, stop. Otherwise, there is some vector $\vec{b}$ that is in the basis set B, that is not in $N_B \cup R_B$. (Again, this is because if not, T would contain the entirety of a basis for U in its span, which would force it to contain U itself in its span!)

   (b) Put $\vec{b}$ in R_B. Notice that if $N_B \cup R_B$ was linearly independent before we added $\vec{b}$, it is still linearly independent, because $\vec{r}$ was not in the span of $N_B \cup R_B$.

   As before, this process will eventually end, because we only have finitely many elements in B.

3. At the end of this process, we have two sets R_B, N_B such that their union is a basis for U. Look at the set $T(R_B) = \{T(\vec{r}) \mid \vec{r} \in R_B\}$. We claim that $T(R_B)$ is a basis for the range of T. To prove this, we will simply show that this set spans the range, and is linearly independent.

4. To see that $T(R_B)$ spans the range: take any $\vec{v}$ in the range of T. Because $\vec{v}$ is in the range, there is some $\vec{u} \in U$ such that $T(\vec{u}) = \vec{v}$. Write $\vec{u}$ as some linear combination

of elements in our basis $R_B \cup N_B$: i.e.

$$\vec{u} = \sum_{\vec{r_i} \in R_B} \lambda_i \vec{r_i} + \sum_{\vec{n_i} \in N_B} \gamma_i \vec{n_i}.$$

Let $\vec{w}$ be the following vector:

$$\vec{w} = \sum_{\vec{r_i} \in R_B} \lambda_i \vec{r_i}$$

In other words, $\vec{w}$ is just $\vec{u}$ if you get rid of all of the parts made using vectors in the null space! Now, simply observe that on one hand, $\vec{w}$ is an element in the span of R_B, and on the other

$$\begin{aligned}
\vec{v} = T(\vec{u}) &= T\left(\sum_{\vec{r_i} \in R_B} \lambda_i \vec{r_i} + \sum_{\vec{n_i} \in N_B} \gamma_i \vec{n_i}\right) \\
&= T\left(\sum_{\vec{r_i} \in R_B} \lambda_i \vec{r_i}\right) + T\left(\sum_{\vec{n_i} \in N_B} \gamma_i \vec{n_i}\right) \\
&= \sum_{\vec{r_i} \in R_B} \lambda_i T(\vec{r_i}) + \sum_{\vec{n_i} \in N_B} \lambda_i T(\vec{n_i}) \\
&= \sum_{\vec{r_i} \in R_B} \lambda_i T(\vec{r_i}) + \vec{0} \\
&= \sum_{\vec{r_i} \in R_B} \lambda_i T(\vec{r_i}),
\end{aligned}$$

because things in the null space get mapped to $\vec{0}$!

Therefore, given any $\vec{v}$ in the range, we have expressed it as a linear combination of elements in $T(R_B)$. In other words, $T(R_B)$ spans the range of V.

5. To show that $T(R_B)$ is linearly independent: take any linear combination of vectors in $T(R_B)$ that is equal to $\vec{0}$:

$$\sum_{\vec{r_i} \in R_B} \lambda_i T(\vec{r_i}) = \vec{0}.$$

We want to show that all of the coefficients λ_i are 0.

To see this, start by using the fact that T is linear:

$$\vec{0} = \sum_{\vec{r_i} \in R_B} \lambda_i T(\vec{r_i}) = T\left(\sum_{\vec{r_i} \in R_B} \lambda_i \vec{r_i}\right).$$

This means that the vector

$$\sum_{\vec{r_i} \in R_B} \lambda_i \vec{r_i}$$

is in the null space of T. Therefore, we can create a linear combination of vectors in N_B that are equal to this vector: i.e. we can find elements in N_B and constants γ_i such that

$$\sum_{\vec{r_i} \in R_B} \lambda_i \vec{r_i} = \sum_{\vec{n_i} \in N_B} \gamma_i \vec{n_i}.$$

But this means that we have

$$\sum_{\vec{r_i} \in R_B} \lambda_i \vec{r_i} - \sum_{\vec{n_i} \in N_B} \gamma_i \vec{n_i} = \vec{0},$$

which is a linear combination of elements in our basis $R_B \cup N_B$ that equals $\vec{0}$! Because this set is a basis, it is linearly independent; therefore all of the coefficients in this linear combination must be 0. In particular, this means that all of the λ_i's are 0, which proves what we wanted: that the set $T(R_B)$ is linearly independent!

Consequently, we have created a basis for U of the form N_B and R_B, where $T(R_B)$ is a basis for the range and N_B is a basis for the null space. This means that, in particular, the dimension of U is the number of elements in N_B (i.e. the dimension of the null space) plus the number of elements in R_B (i.e. the dimension of the range.) So we're done! $\square$